Multidimensional Inequality Comparisons : a Compensation Perspective*

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April 22, 2003

Abstract

The aim of the paper is to provide criterions based on sound ethical grounds for inequality and welfare comparisons in a multi-dimensioned setting. We provide a unified treatment of multi-dimensional egalitarianism and of welfare analysis with needs using a compensation principle and encompassing the case of continuous and discrete distributions. Among the attributes of the individual utility, at least one, (in most cases the current period income) can be used to compensate attributes like past-income, income of the past generation, health, education, needs due to family size. Post-Rawlsian distributive justice (Dworkin, Roemer) argues for compensating an attribute, provided that persons should not be responsible for it. The main theorem exhibits two necessary and sufficient second degree stochastic dominance conditions for the comparison of bivariate distributions. In the case of a discrete compensated variable, the distributions of the compensating variable have to satisfy a condition which degenerates to the Sequential Generalized Lorenz test in case of identical marginal distributions of the compensating variable. Moreover, the distributions of the compensated variable must satisfy the Generalized Lorenz test. Extensions to the case of a trivariate distribution are provided, where we single out three configurations, the full compensation the chain compensation and the single compensation.

Keywords: Multi-dimensioned Welfare, Compensation, Dominance, Lorenz Criterion.

JEL Codes: D3, D63, I31

1 Introduction

Inequality among a group of people, has often been measured in terms of income (e.g. Kolm (1969), Atkinson (1970), Sen (1973)). However, social scientists and economists (Sen (1987), (1992)) have argued that income is not a sufficient statistic for welfare and should be supplemented by other attributes of well-being such as health, education, literacy. Income varies

*We are grateful to the British Academy and to the CNRS for the grant n° XXX that made this project possible. The first author acknowledges the ESRC grant n° $R000$. The second author thanks the financial support of the European Commission through the Contract ERBFMRXCT980248. We thank Jean-Luc Prigent for some helpful comments. The usual disclaimer applies.

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over time and comparing inequality of intertemporal income streams provides another example of a multidimensional framework. Among the attributes, the income or the education of the parents might also be considered for inequality comparisons among dynasties. Last but not least, households are the elementary statistical units in many data sets, which implies accounting for differences in family size and in other household characteristics for welfare comparisons among these statistical units.

The derivation of social dominance conditions for multidimensioned welfare analysis is arguably one of the main challenges of modern welfare analysis. The multi-dimensioned welfare literature appears in two veins. In the first one, called hereafter the multi-dimensioned one, which can be traced back to Kolm's paper (1977), all attributes have symmetric roles. Multidimensional dominance criteria consist in seeking unanimity among large classes of social welfare functions over the ranking of allocations. A bunch of papers have been devoted to this topic (Huang, Kira and Vertinsky (1978), Marshall and Olkin (1979, Chapter 15), Atkinson and Bourguignon (1982), Le Breton (1986), Bourguignon (1989), Koshevoy (1995, 1998) and Koshevoy and Mosler (1996)). In particular, Atkinson and Bourguignon (1982), below denoted AB1, propose dominance relationships for various classes of utility functions defined by the signs of their derivatives up to the fourth order. Nevertheless, it seems fair to say that no simple criterion to check multidimensional dominance has reached popular support among applied economists and even among theorists. This unsuccess stems from the lack of intuitive appeal of some conditions on utility functions. Up to now, one does not dispose of well-accepted normative conditions to anchor multi-variate stochastic dominance analysis, in contrast with the central role of transfer axioms for univariate stochastic dominance.

The landmark article by Atkinson and Bourguignon (1989), below denoted AB2, generates a second type of multi-dimensioned welfare analyses. The attributes are no longer symmetric and the focus is on the measurement of income inequality accounting for households' different needs, generated for instance by different family size. Then, one attribute (e.g., family size) is used to divide the population into homogeneous groups, while social welfare defined from the second attribute (income) is considered within the groups and in the whole society. AB2 have provided a simple and elegant test for making welfare comparisons in such a context : the *Sequential Generalized Lorenz* (SGL*)* quasi-ordering*,* which extends the *Generalized Lorenz* (GL) quasi-ordering (Shorrocks (1983)) to the situation where the population is partitioned into subgroups on the basis of needs. A growing number of papers deal with this "needs approach" and a non exhaustive list would include Bourguignon (1989), Jenkins and Lambert (1993), Shorrocks (1995), Ebert (1995, 1997, 1999, 2000), Chambaz and Maurin (1998), Ok and Lambert (1999), Moyes (1999), Bazen and Moyes (2001), Lambert and Ramos (2002), Fleurbaey, Hagneré and Trannoy (2001a and b).

It is important to notice that in the second approach, the marginal distribution of needs has been neutralized. In the original paper by Atkinson and Bourguignon (1989), the marginal distribution of needs is assumed to be identical in both populations. Nevertheless, Jenkins and Lambert (1993), Chambaz and Maurin (1998) have shown how the SGL test may be extended to the case where distributions of needs differ. Moreover, Moyes (1999) and Bazen and Moyes (2001) have modified the assumption added by Jenkins and Lambert so as to allow the marginal distribution of needs to play a role in performing the comparison between two distributions. Doing so makes the two approaches less distinct and it raises the question of the relevance to consider two distinct approaches.

For an appraisal of a public policy, the rationale to consider these two approaches seems the following. To make the discussion more concrete, assume that the two attributes are income and the health status. Governement intervention may influence the two marginal distributions. For example fiscal policy has a direct impact on the income distribution, while a public health care affects the health status distribution. If one wants to evaluate the sole consequences of the fiscal policy, the needs approach, keeping constant the marginal distribution of health status seems appropriate, but if the impacts of the two policies have to be assessed, the multidimensioned has to be called for. Hence, there is a kind of division of labor between the two approaches but it would be helpful if they were developped in a consistent way. One aim of this paper is to provide such an integration.

The marginal utility function is supposed to be identical across agents with respect to each attribute and positive and decreasing. However, this standard assumption is not sufficient to obtain critera with a sufficient discriminatory power, as shown by AB2 in the needs context.

The basic idea of the paper is to consider that among all the attributes, at least one can be used to make direct transfers between individuals. In the income-health example, it is the income of the current period which plays the role of the compensating variable. The other variables are the compensated variables (income in the past, income of the past generation, health, education, family size and so on). Many recent contributions in distributive justice (see for an overview, Roemer (1996), Fleurbaey (1995)) provide ethical grounds for compensating an attribute. The broad idea is that welfare differences are acceptable when they are due to characteristics for which agents can be deemed responsible. On the contrary, individuals should be compensated for attributes for which they cannot be held responsible. The discussion on the exact location of the cut between the two sets of characteristics is far from closed (Dworkin (1981), Sen (1985,1992), Roemer (1985,1993), Arneson(1989), Cohen(1989)). For instance, Dworkin proposed to include preferences in the former category and resources (including internal one like innate talent) in the latter. Atkinson and Bourguigon (2000), who allude to the possibility of compensation p.46, seem to endorse Dworkin's position : "Differences in innate abilities, needs or handicaps would seem to require some kind of compensation, but not differences in effort, resulting from differences in tastes or preferences". Schokkaert and Devooght (1998) present experimental results from simple survey questions suggesting that this broad idea of compensation for "uncontrollable" factors find some echo from a majority of a sample of respondents. Still, the reader does not have to agree with these philosophical premices to accept the validity of our results. It is sufficient for us that some compensation can be defended on some grounds, whatever they are, exogeneity of some attributes or other reasons such as basic egalitarianism.

In a one dimension setting, it is well known that the two statements (a) and (b) are equivalent: (a) 'Pigou-Dalton's transfers improve welfare'; (b) 'The utility function which intervenes in an additively separable welfare function is concave'. In the same vein, we capture the idea that compensation is good for social welfare, meaning that transfering income from an healthy to an handicapped person having the same level of income, is recommanded, by imposing a negative sign on the cross derivative of the utility function between the compensating variable and the compensated one. In others words, the marginal utility of income is decreasing with the level of the compensated variable. For instance the healthier you are, the lower your claims are to a redistribution, other things being equal.

Performing compensation seems all the more appropriate that handicapped people often belong to the bottom part of the income distribution. A wealthy person does not seem to be a good candidate for public funds, even if she suffers from some disadvantage. We supplement our welfare analysis by an additional assumption trying to catch this intuition: the decrease in the marginal utility of income with the level of the compensated variable is all the smaller when the agent is rich. For instance, the differences in marginal utilities of income between a healthy rich person and an ill rich person may be tiny, implying that the second one does not deserve to be compensated for his bad health.

The reader aware of the literature will recognize that these assumptions are akin to assumptions made by AB2 in a context where the compensated variable is discrete. In other words, our analysis plugs AB2 assumptions into AB1 framework. How surprising this might appear, such an approach has not been pursued so far, to the best of our knowledge. The obtained criterion would not come to a surprise to specialists, but it provides a useful test which encompasses the needs approach in the multi-dimensioned one. If a distribution of attributes dominates another one then two conditions are fulfilled. First, in the case of a discrete compensated attribute, the distributions of the compensating variable meet a Sequential Restricted Generalized Lorenz test (SRGL), a test linked to the SGL test. When the marginal distribution of the compensating variable is considered as fixed, the SRGL test degenerates to the SGL test. Second, the distribution of the compensated variable satisfies the GL test. If we take again the income-health example, to get dominance, namely an improvement in social welfare in moving from joint-distribution 1 to joint-distribution 2, the income distribution 1 must SRGL-dominates income distribution 2 and the health distribution 1 must GL-dominates income distribution 2.

Such a criterion provides a simple test of welfare improvements in a multidimensional setting, with two additional advantages. First it is in tune with the criterion obtained in the needs analysis and second it corresponds to dominance for a class of utilities functions that have ethical and intuitive meaning. Moreover, such a criterion can be extended to variants of ethical conditions (for example some transfer sensitivity properties), or to the case of more than two attributes.

The paper is organized as follows. The next section provides the main result with two attributes and gives the conditions to verify in terms of second stochastic dominance conditions or inverse stochastic conditions. A comparison with results obtained in the literature is provided and transfer sensitivity as defined by Shorrocks and Foster (1987) is also introduced. In Section 3, two compensated variables are supposed to matter for social welfare. Finally, Section 4 concludes. All proofs of propositions are gathered in the Appendix.

2 Two Goods Case Result

Let us consider the bivariate distribution of a random variable $X = (X_1, X_2)$ where by convention subscript 1 is used for the compensating good and 2 for the compensated one. We assume that the support of X is the rectangle $[0, a_1] \times [0, a_2] = A_1 \times A_2$ where a_1 and a_2 are in $\mathbb{R}_+$. We denote the corresponding joint cumulative distribution function by $F(x_1, x_2)$ and by F_1 and F_2 the respective marginal cdf of X_1 and X_2. F_i $(i = 1, 2)$ are any positive increasing and right-continuous functions with range $[0, 1]$. It is important to notice that we do not assume $F(x_1, x_2)$ to be continuous or discrete and all theorems are proved for the most general case. By Ji$\widetilde{r}$ina's theorem, (see for instance Métivier p.142), there exists a conditional cumulative distribution function of X_1 with respect to X_2 denoted F_1^2 such that for any $(x_1, x_2) \in A_1 \times A_2$,

$$F(x_1, x_2) = \int_{[0,x_2]} F_1^2(x_1 | X_2 = t) dF_2(t). \tag{1}$$

Another joint cumulative distribution function is denoted F^*. Let $U(x_1, x_2)$ be the utility function which can convey private values as well as social ones. It is assumed to be Lebesgue integrable with respect to F and F^*. The social welfare function associated to F is assumed to be additively separable and is computed as

$$W_F := \int_{A_1 \times A_2} U(x_1, x_2) dF(x_1, x_2))$$

or using the decomposition expressed by (1) this may be written

$$W_F = \int_{A_2} \left[\int_{A_1} U(x_1, x_2) dF_1^2(x_1 | X_2 = x_2) \right] dF_2(dx_2) \tag{2}$$

where the marginal distribution according to X_2 appears distinctly. The inner expression in (2) is the welfare of the subpopulation of individuals having in common the same amount of good 2 and the total welfare is the sum of these subpopulations welfares over x_2. This expression of welfare generalizes AB2' s one (expression 12.3 p.353) for which F_2 is supposed to be discrete and F_1^2 continuous.

The change in welfare between two distributions F and F^* is given by

$$\triangle W_U := W_F - W_{F^*} = \int_{A_1 \times A_2} U(x_1, x_2) \triangle dF x_1, x_2))$$

where $\triangle F$ denotes $F - F^*$.

Dominance in the welfare literature is usually defined as unanimity for a family of social welfare functions based on a set of specific utility functions.

Definition 2.1 *F dominates F^* for a family $\mathcal{U}$ of utility functions if and only if $\triangle W_U \geq 0$ for all utility functions U in $\mathcal{U}$. This is denoted $F D_{\mathcal{U}}\ F^*$.*

U is assumed to be continuously differentiable to the required degree. The partials with respect to each variable are denoted by subscripts.

2.1 A central result

We start with the set $\mathcal{U}^2$ of increasing utility functions concave in each of their arguments and respecting the following signs of the partials:

$$\mathcal{U}^2 = \{U_1, U_2 \geq 0, U_{11} \leq 0, U_{22} \leq 0,\ U_{12} \leq 0, U_{121} \geq 0\}^1. \tag{3}$$

Another way to describe this set is to introduce the definition of a non-increasing increments utility function.

A utility function is said to have non-increasing increments if

$$u(x + h) - u(x) \geq u(y + h) - u(y)$$

for all $x, y \in \mathbb{R}_+^\ell$ such that $x \leq y$, and for all $h \in \mathbb{R}_+^\ell$. When u is twice continuously differentiable on $\mathbb{R}_+^\ell$, then u has non-increasing increments if and only if $u''_{jk} \leq 0\ \forall j, k \in L$, a condition known under the label of ALEP substituability[2] (see Chipman (1977)). When

[1] The class $\mathcal{U}^1$ will be introduced later on in the text.

[2] ALEP stands for Auspitz-Lieben-Edgeworth-Pareto.

a person gets more affluent in each dimension, marginal utility is required to decrease in each dimension. Then we can describe the set $\mathcal{U}^2$as the set of increasing, ALEP substituable utility functions respecting moreover $U_{121} \geq 0$.

We offer three illustrations which may support the negativity of the cross second-order partial as a reasonable requirement on the social evaluation function in some circumstances and the other signs of the partials as well.

Example 2.1 *Handicap to individual well-being*

The first example is when the opposite of the second attribute is seen as an handicap to the individual well-being. A bad health, a true handicap comes to mind as natural examples.

Example 2.2 *Mobility measurement*

The second example is in the domain of mobility measurement. Suppose that we look for dominance conditions under which we are able to rank mobility processes between two generations, the sons'one and the fathers'one: the first attribute stands for the rank of the son in the son's income distribution while the second attribute figures out the rank of the father in the father's income distribution. $F(x_1, x_2)$ gives the proportion of couples son-father, such that son gets a rank at most equal to x_1, while father's rank was at most equal to x_2. Atkinson (1981) provides conditions under which the social evaluation function can be described in an additive way with a negative sign for the cross second-order partials of the "utility function" with the two ranks as attributes.

Example 2.3 *Family size*

Differences in family size (n) is one of the favourite example of differences in need. Suppose that attribute 2 is the deviation to some maximal family size $\overline{n}$, $i.e$, $x_2 = \overline{n} - n$, while the first attribute is household income (y). We investigate at which conditions $U(x_1, x_2)$ belongs to $\mathcal{U}^2$ or equivalently a household utility function $u(y, n)$, where family size is treated as a real variable for convenience, satisfies $u_y \geq 0, u_{yy} \leq 0, u_n \leq 0, u_{nn} \leq 0,$ $u_{yn} \geq 0, u_{yny} \leq 0,$ namely, a child counts as a social cost and not as as a social benefit.

They are many ways to deal with such a question. First consider the common practice of equivalizing income. When a particular equivalence scale function $e(n)$ is chosen, social welfare can be computed by aggregating the utility levels of equivalent incomes defined as $\frac{y}{e_{(n)}}$over the population. In this framework, Ebert (1999) proposed to adopt the following household utility function:

$$u(y, n) = e(n)v\left(\frac{y}{e_{(n)}}\right). \text{ with } e'(n) \geq 0$$

Assuming $v' \geq 0$ and $v'' \leq 0$, it is readily shown that it ensures $u_y \geq 0, u_{yy} \leq 0$ and $u_{yn} \geq 0$. Yielding $u_{yny} \leq 0$ is more demanding and requires that $v''' \geq 0$, and a little bit more, precisely that the elasticity of v'' with respect to equivalent income must be larger than 1 in absolute terms. An isoelastic function with respect to income

$$v(y) = \frac{1}{1-\beta}x^{1-\beta}, \qquad \text{with } 0 \leq \beta < 1 \tag{4}$$

satisfies the above conditions, (the elasticity of $v''$is equal to $1+\beta$) but anybody will recognize that they are unduly restrictive. Now,

$$u_n = e'(n)(v(\frac{y}{e(n)}))(1-\epsilon)$$

with $\varepsilon = \frac{y}{e_{(n)}} v'(\frac{y}{e_{(n)}})$, the elasticity of v with respect to equivalent income. Assuming either v negative, which is immaterial, and $\epsilon \leq 1$ or v positive and $\epsilon \geq 1$ provide the requested sign, i-e, an additional member is viewed as a cost. Finally,

$$u_{nn} = e''(n)(v(\frac{y}{e_{(n)}})(1-\epsilon) + \frac{y^2}{(e(n))^3} v''(\frac{y}{e_{(n)}}). \tag{5}$$

In the case where $e''(n) \geq 0$[3], the conditions ensuring the negativity of u_n guarantees the negativity of u_{nn} as well. It is worth it to mention that a parametric equivalence specification proposed by Banks and Johnson (1994), $e(n) = n^\theta$ with $0 < \theta \leq 1$, does not respect the requirement of the convexity of the equivalence function with respect to family size. In summary, a linear equivalence scale and an isoelastic ffunction with respect to income with an elasticiy smaller than 1 does the job.

Not everybody is pleased with the concept of an equivalence scale and one maysprefer a more structural approach where the allocation of the family budget among its member is explicitely allowed for. Here we pursue an approach suggested by Bourguignon (1989) who investigated what are the properties of the indirect household utility function in a Samuelson' s model of the family in presence of public goods. It is assumed that each individual is endowed with the same continuous, increasing and quasi-concave utility function V defined on two attributes, a private good x and a good g which is public within the family. Assume that each family of size n behaves like a utilitarian society[4]. It allocates the household budget y such

$$\max_{x,g} nV(x,g) \ / \ nx + g = y. \tag{6}$$

The FOC is given by

$$n(V_x - nV_g) = 0.$$

Let x^* be the optimal solution. We assume that the private good is normal which requires $-V_{xg} + nV_{gg} < 0$. It turns out that this condition implies $x_n^* \leq 0$ as well. Introducing the demand functions $x(y,n)$ and $g(y,n)$ associated to the above maximization program, we define the indirect utility function

$$u(y,n) = nV(x(y,n), g(y,n))$$

Bourguignon claimed that the condition $u_{yn} \geq 0$ is not ensured and depends upon the elasticity of substitution between private and public consumption (e.g., note 2 p.71). The computations of the derivatives of the indirect utility functions yield

$$u_y = nV_g \geq 0$$

[3]The opposite case, $e''(n) < 0$, is a dead end since the first term in 5 will be positive.

[4]The same reasoning holds if we only assume that the household allocate goods in an efficient way. For the problem in touch, it is easier to consider that individuals (with the same utility function) are treated in a symmetric way.

$$u_{yy} = n\frac{V_{gg}V_{xx} - V_{xg}^2}{V_{xx} - 2nV_{xg} + n^2V_{gg}} \leq 0$$

thanks to the quasi-concavity assumption;

$$u_n = V - nx^*V_g \leq 0$$

if $V < 0$ which is immaterial;

$$u_{nn} = -x^*(2+\eta_n)V_g - nx^*x_n^*V_{xg} + x^*(1+\eta_n)V_{gg}$$

with $\eta_n = \frac{nx_n^*}{x^*}$ the elasticity of the individual private consumption to family size,

$$u_{yn} = V_g + nV_{gx}x_n^* - V_{gg}x^*(1+\eta_n).$$

If $\eta_n > -1$,and $V_{gx} < 0$ then $u_{yn} \geq 0$ and $u_{nn} \leq 0$.Therefore, provided that we admit that an increase of family size of 10% does not decrease the individual consumption of more than 10% and provided that public good and private consumption are ALEP substitutes, we get all the needed signs for the first order and second order partials. Intuition seems lost when we investigate the conditions under which we may obtain the required sign for u_{yny}. In conclusion of this example related to family size, the signs of the partials involved in the $\mathcal{U}^2$-class seem microbased but the sign of the third-order partial,

In order to present the result which expresses the conditions to be satisfied in terms of second-degree stochastic dominance, it is convenient to define

$$H_i(x_i) = \int_0^{x_i} F_i(s)ds, \; i = 1, 2$$

and

$$H_1(x_1; x_2) = \int_0^{x_1} F(s, x_2)ds. \tag{7}$$

In that case, we obtain the following result of multi-variate stochastic dominance.

Proposition 2.1 *Let F and F^* two cdfs.*

$$FD_{\mathcal{U}^2}F^* \tag{A1}$$

$$\Updownarrow$$

$$\triangle H_2(x_2) \leq 0, \quad \forall x_2 \in X_2 \tag{B}$$

$$\triangle H_1(x_1; x_2) \; \leq 0, \; \forall x_2 \in X_2, \; \forall x_1 \in X_1 \tag{C}$$

(B) is the standard second degree stochastic dominance expression for variable 2, while (C) is a mixed second degree stochastic dominance term, where we integrate the cdf of the joint distribution with respect to variable 1. In particular, (C) implies second degree stochastic dominance for variable 1 as well.

Remark 2.1 *If we consider the twin class of $\mathcal{U}^2$,$\mathcal{U}^{2*} = \{U_1 \geq 0, U_2 \leq 0, U_{11} \leq 0, U_{22} \geq 0, \; U_{12} \leq 0, U_{121} \geq 0\}$ where the second attribute appears as a bad with a desutility increasing and convex, it is sufficient to modify condition B in $\triangle H_2(x_2) \geq 0$ to obtain a dominance result for the $\mathcal{U}^{2*}$ class.*

We can compare our result with the most significant results obtained by AB1. Our class is clearly intermediate between the first class they are interested in defined as follows:

$$\mathcal{U}^{AB1_1} = \{U_1, U_2 \geq 0, U_{12} \leq 0\}. \tag{8}$$

which has a flavour of first degree stochastic dominance and the second class given by

$$\mathcal{U}^{AB1_2} = \{U_1, U_2 \geq 0, U_{11} \leq 0, U_{22} \leq 0, U_{12} \leq 0, U_{121}, U_{212} \ \geq 0, U_{1122} \leq 0\}. \tag{9}$$

The condition U_{221} may be interpreted in terms of transfer sensivity (see the set $\mathcal{U}^4$ below) but, more importantly, put together with $U_{21} \leq 0$, it supports the idea that attribute 2 can compensate for deficiencies in attribute 1. $\mathcal{U}^{AB1_1}$ or $\mathcal{U}^{AB1_2}$ treats symetrically the two variables while there is an asymmetry in the way they are considered in $\mathcal{U}^2$. For the sake of completeness, we remind the results obtained by AB1.

Theorem 2.2 *Let F and F^* two cdfs admitting a density.*

$$FD_{\mathcal{U}^{AB1_1}} F^* \tag{A}$$

$$\Updownarrow$$

$$\Delta F(x_1, x_2) \leq 0, \quad \forall x_1 \in X_1, \forall x_2 \in X_2 \tag{D$_L$}$$

$$FD_{\mathcal{U}^{AB1_2}} F^* \tag{A}$$

$$\Updownarrow$$

$$\triangle H_1(x_1) \leq 0\,, \quad \forall x_1 \in X_1 \tag{10}$$

$$\triangle H_2(x_2) \leq 0\,, \quad \forall x_2 \in X_2 \tag{11}$$

$$\Delta H(x_1, x_2) \leq 0, \quad \forall x_1 \in X_1, \forall x_2 \in X_2 \tag{D$_L$}$$

Our criterion leads to a less partial quasi-ordering of bidimensional distributions than their first criterion but to a more partial ordering than their second criterion.

Expressing conditions of stochastic dominance in terms of Lorenz curves make them more palatable for scientists in the fields of inequality measurement as Atkinson's understood it more than thirty years ago (Atkinson (1970)).

2.2 Inverse Stochastic dominance results

Let us define the right-inverse of a positive increasing and right-continuous function $F(x)$ with p the image in $[0,1]$ such that

$$F^{-1}(p) = \sup_{F(x) \leq p} x.$$

The generalized Lorenz (GL) curve (see Shorrocks (1983)) of the marginal cdf F_i for $i = 1, 2$, $\mathcal{L}_{F_i}(p)$ is defined on $[0,1]$ by

$$\mathcal{L}_{F_i}(p) = \int_0^p F_i^{-1}(t)dt. \tag{12}$$

In case where variable 1 represents income, the Generalized Lorenz curve of F_1 shows the cumulative total income received by the proportion p of the population, individuals being ranked increasingly according to their income.

We also need to define a new concept, the restricted generalized Lorenz (RGL) curve for a given value of X_2. For each x_2 in X_2, the function F_{x_2} is defined on X_1 by the equation

$$F_{x_2}(x_1) = F(x_1, x_2)$$

For a given x_2, $F_{x_2}(x_1)$ is at most equal to $F(a_1, x_2) = F_2(x_2)$. Let now define the right inverse as

$$\forall p \in [0, F_2(x_2)],\ F_{x_2}^{-1}(p) = \sup_{F_{x_2}(x_1) \leq p} x_1.$$

The RGL curve for the joint cdf on $[0, F_2(x_2)]$ is described by

$$\mathcal{C}_{F_{x_2}}(p) = \int_0^p F_{x_2}^{-1}(t)dt \tag{13}$$

and under the same proviso it gives the cumulative income received by the proportion p of the population having at most a level of the compensated variable equal to x_2.

Proposition 2.3 *Let F and F^* two cdfs.*

$$\triangle H_2(x_2) \leq 0\,, \quad \forall x_2 \in X_2 \ \Leftrightarrow \mathcal{L}_{F_2}(p) \geq \mathcal{L}_{F_2^*}(p),\ \forall p \in [0,1] \tag{A$_L$}$$

$$\forall x_2 \in X_2, \left[\triangle H_1(x_1; x_2) \leq 0,\ \forall x_1 \in X_1 \Rightarrow \mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F^*_{x_2}}(p),\ \forall p \in [0, \min(F_2(x_2), F_2^*(x_2))]\right] \tag{B$_L$}$$

$$\forall x_2 \in X_2 | F_2(x_2) \leq F_2^*(x_2), \left[\triangle H_1(x_1; x_2)\ \ \leq 0,\ \forall x_1 \in X_1 \Leftrightarrow \mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F^*_{x_2}}(p),\ \forall p \in [0, F_2(x_2)]\right] \tag{C$_L$}$$

We do not get a complete set of conditions in terms of inverse stochastic dominance equivalent to conditions obtained in Proposition 1. Nevertheless, we get a set of equivalence and necessary conditions which can be proved to be useful in applied works. First of all, dominance in term of the GL curve of the compensated variable is required to get dominance for the class of utility functions considered. Second, dominance of the RGL curve of the compensating variable for any value of the compensating variable, is also necessary on the domain given by the intersection of the domains of definition of the two RGL curves. The RGL-test expressed in condition B$_L$ or C$_L$ must be performed in a sequential way (SRGL-test).

For the sake of illustration, consider the case of a discrete compensated variable, namely, F_2 and F_2^* are two step functions with jumps at $x_{21}, ..., x_{2k}$ for F_2 and at $x_{21}^*, ..., x_{2l}^*$ for F_2^*. Condition C$_L$ indicates that nothing is required for all x_2 strictly smaller than $\max(x_{21}, x_{21}^*)$. For $x_2 = \max(x_{21}, x_{21}^*)$, compute $F_2(.)$ and $F_2^*(.)$ and look at the RGL-curves of the compensating variable for the subpopulation having a value of the compensated variable smaller than or equal to $\max(x_{21}$,$x_{21}^*)$. This curve for the dominating distribution must be above to that of the dominated distribution for all cumulated proportions of population smaller than or equal to the minimum of $F_2(\max(x_{21}, x_{21}^*))$ and of $F_2^*(\max(x_{21}, x_{21}^*))$. The first sequential checking is done. If it is positive, then go to the next step where we restrict our attention to the values of $x_{22}, ..., x_{2k}$ and of $x_{22}^*, ..., x_{2l}^*$ strictly larger than $\max(x_{21}, x_{21}^*)$ and we consider the minimum of these values. Call it m_1. The RGL-test has to be performed for $x_2 = m_1$. If it is positive, turn to the third sequential checking. Let consider the values of $x_{22}, ..., x_{2k}$ and of $x_{22}^*, ..., x_{2l}^*$ strictly larger than m_1 and consider the minimum of these values. Call it m_2 and so on and so forth, up to the last sequential checking which occurs for $x_2 = \max(x_{2k}, x^*_{2l})$. Here, the comparison is tantamount to perform the classic GL test for the compensating variable, since $F_2(\max(x_{2k}, x^*_{2l})) = F^*_2(\max(x_{2k}, x^*_{2l})) = 1$. It turns out that the SRGL-test is only sufficient when the distribution of the compensating variable for F dominates its counterpart for F^* to the first order. When it is not the case, it means that there exists values of the compensating variable such that the proportion of individuals having at most this value is larger in F than in F^*. Condition B implies that it cannot happen for the smallest value of x_2 . It can also be observed that

$$\forall x_2 \in X_2 | F^*_2(x_2) \leq F_2(x_2),$$
$$\left[\triangle H_1(x_1; x_2) \leq 0, \, \forall x_1 \in [0, F_2^{-1}(F^*_2(x_2)] \Leftrightarrow \mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F^*_{x_2}}(p), \, \forall p \in [0, F^*_2(x_2)] \right].$$

So roughly speaking, checking the RGL-condition is sufficient in the the bottom part of the joint distribution but not in the middle or top part.

The interest of obtaining an equivalence in terms of concentration curves of second stochastic dominance term such as $\triangle H_1(x_1; x_2)$ goes beyond the framework of this article and allows to interpret the results achieved by authors like Jenkins and Lambert (1993), Chambaz and Maurin (1998), Lambert and Ramos (2002), and Fleurbaey *et al.*(2001a) who considered the need approach when the marginal distribution of needs differ between the two distributions. For instance, for the family of utility functions considered by Moyes (1999) and Bazen and Moyes (2001), one gets a complete characterization result using the concept of RGL-curve. These authors considered the following set of utility functions.

$$\text{Let } \mathcal{U}^1 = \{U_1, U_2 \geq 0, U_{11} \leq 0, U_{12} \leq 0, U_{121} \geq 0\}. \tag{14}$$

The marginal utility with respect to the compensating variable is no more required to be decreasing. Proposition 2 states that the SRGL test is only sufficient when the distribution of the compensating variable for F dominates its corresponding distribution $F^*$to the first order. In that case, a necessary and a sufficient condition involving the RGL-curve is obtained.

Corollary 2.4 *Let F and F^* two cdfs.*

$$F D_{\mathcal{U}^1} F^* \tag{A$_0$}$$
$$\Updownarrow$$
$$\triangle F_2(x_2) \leq 0, \quad \forall x_2 \in X_2 \tag{B$_0$}$$
$$\mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F^*_{x_2}}(p), \quad \forall p \in [0, F_2(x_2)], \forall x_2 \in X_2 \tag{C$_0$}$$

Proof. Using the proof of proposition 1, it is readily shown that A_0 is equivalent to B_0 and C. For proof of sufficiency, see equation 29 in the Appendix. Equivalence C_L in Proposition 2 shows that in presence of B_0, C is equivalent to C_0. ■

Combining Proposition 2 and Corollary 1, we find both a sufficient condition and a necessary one to check dominance for the family $\mathcal{U}^2$.

Corollary 2.5 *Let F and F^* two cdfs.*

$$\triangle F_2(x_2) \leq 0, \quad \forall x_2 \in X_2 \text{ and } \mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F^*_{x_2}}(p), \quad \forall p \in [0, F_2(x_2)], \forall x_2 \in X_2 \tag{B$_0$\&C$_0$}$$
$$\Downarrow$$
$$F D_{\mathcal{U}^1} F^* \tag{A}$$
$$\Downarrow$$
$$\mathcal{L}_{F_2}(p) \geq \mathcal{L}_{F^*_2}(p), \quad \forall p \in [0, 1] \text{ and } \mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F^*_{x_2}}(p) \quad \forall p \in [0, \min(F_2(x_2), F^*_2(x_2)], \forall x_2 \in X_2, \tag{A$_L$\&B$_L$}$$

The link of the SRGL criterion with the SGL one pionnered by AB2 will become apparent in the particular case of identical marginal distributions of the compensating variable. Let define for a given x_2 and for any $x_1 \in X_1$

$$G_{x_2}(x_1) = \frac{F_{x_2}(x_1)}{F_2(x_2)}$$

and for any $p \in [0,1]$

$$G_{x_2}^{-1}(p) = \sup_{G_{x_2}(x_1) \leq p} x_1.$$

The GL-curve corresponding to the subpopulation for which the value of the compensated variable is at most x_2 is defined by

$$\forall p \in [0,1], \quad \mathcal{L}_{F_{x_2}}(p) = \int_0^p G_{x_2}^{-1}(t)dt. \tag{15}$$

If the marginal distributions of the compensating variable are identical, then our criterion boils down to the SGL one and we are back to the "need approach" considered by AB2. In this particular setting, we have extended their result to the case of a continuous distribution of needs.

Corollary 2.6 *Let F and F^* two cdfs such that $F_2 \equiv F_2^*$.*

$$F D_{\mathcal{U}^1} F^* \tag{A}$$

$$\Updownarrow$$

$$\mathcal{L}_{F_{x_2}}(p) \geq \mathcal{L}_{F_{x_2}}(p)\,, \quad \forall p \in [0,1], \forall x_2 \in X_2 \tag{D$_L$}$$

Proof. Condition B_L in proposition 2 is satisfied by assumption. Condition C_L becomes:$\forall x_2 \in X_2, \mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F^*_{x_2}}(p) \;\; \forall p \in [0, F_2(x_2]$, which is indeed equivalent to condition D_L. ■

2.3 Introducing transfer sensivity

We have already introduced conditions on the signs of the third cross-partial. A point worth noting about the that four signs of third partial derivatives of the utility function resume all the information at this stage, $U_{111}, U_{222}, U_{121}, U_{212}$. In the litterature, there has been some interest in posing the sign of the direct third-order partials. Transfer sensitivity (see Shorrocks and Foster (1987) for the general study of transfer sensivity and Lambert and Ramos (2001) for an application to the needs approach), roughly speaking, means that the planner is more sensible to transfers performed at the bottom of the distribution than at the top. Transfer sensitivity is equivalent to requiring the positivity of the third partial derivative. One may be concerned by imposing transfer sensivity to either the marginal distribution of the compensating variable or the marginal distribution of the compensated variable. Consequently, we define the following sets of utility functions.

$$\text{Let } \mathcal{U}^3 = \{U_1, U_2 \geq 0, U_{11} \leq 0, U_{222} \geq 0, U_{22} \leq 0, \;\; U_{12} \leq 0, U_{121} \geq 0\} \tag{16}$$

$$\text{Let } \mathcal{U}^4 = \{U_1, U_2 \geq 0, U_{11} \leq 0, U_{111} \geq 0, U_{22} \leq 0, \;\; U_{12} \leq 0, U_{121} \geq 0\} \tag{17}$$

We need to define the third degree stochastic term for the marginal distributions $L_i(x_i) = \int_0^{x_i} H_i(s)ds, \;\; i = 1, 2$. Then, we obtain the following equivalence.

Proposition 2.7 *Let F and F^* two cdfs. (i)$F\ D^*_{\mathcal{U}^4}\ F^* \Leftrightarrow F D_{\mathcal{U}^2} F^*$ and (ii)*

$$f\ D_{\mathcal{U}^3}\ f^* \tag{A3}$$

$$\Updownarrow$$

$$\triangle H_2(a_2) \leq 0 \tag{B$_3$}$$

$$\triangle H_1(x_1;x_2)\ \leq 0\,,\ \forall x_2 \in X_2,\ \forall x_1 \in X_1 \tag{C}$$

$$\triangle L_2(x_2)\ \leq 0, \forall x_2 \in X_2 \tag{D$_3$}$$

Thus, requiring transfer sensitivity with respect to the marginal distribution of the compensating variable does not help to obtain a less partial quasi-ordering of distributions than that corresponding to $\mathcal{U}^2$ class. On the opposite, we obtain less stringent conditions of dominance when imposing transfer sensitivity with respect to the marginal distribution of the compensated variable. Condition D_3 is the standard condition of third degree stochastic dominance applied to the marginal distribution of the compensated variable and, as usual, it must be supplemented by a terminal condition of second degree stochastic dominance (Condition B_3) which means that the average of the compensating variable is larger for the the dominant distribution than the one for the dominated distribution. Looking to the three families $\mathcal{U}^1$,$\mathcal{U}^2$,$\mathcal{U}^3$ which correspond respectively to a first, second and third degree perspective on the compensated variable, we yield a less and less demanding criterion and thus a less and less partial quasi-ordering.

We finally define the following set of utility functions.

$$\text{Let } \mathcal{U}^5 = \{U_1, U_2 \geq 0, U_{11} \leq 0, U_{122} \geq 0, U_{22} \leq 0,\ U_{12} \leq 0, U_{121} \geq 0\} \tag{18}$$

The additional assumption ($U_{122} \geq 0$) must be put together with the assumption that $U_{12} \leq 0$ to be interpreted correctly. We already now that marginal utility of income is decreasing with the level of the second attribute. Now we impose that it decreases with a decreasing rate relative to the level of the compensated variable. Since the handicap or the need is just the opposite of the second attribute, the compensation is all the more required that the handicap is strong, i.e. that the level of the second attribute is low. As shown by the next proposition, it turns out that this additional condition does not change the criterion of implementation of dominance.

Proposition 2.8 *Let F and F^* two cdfs. $F D_{\mathcal{U}^4}\ F^* \Leftrightarrow F D_{\mathcal{U}^2}\ F^*$*

3 Three Goods Case Results

Trying to generalize the two goods case result comes as a natural extension. However, it is not a trivial one and the consideration of the three good case sheds light on the difficulties of the exercice. It reveals the natural complexity of any generalization, while indicating the road for promising results.

A trivariate distribution is figured out by a random variable $X = (X_1, X_2, X_3)$ whose joint associated cdf is denoted F on $\mathbb{R}^2_+$ with finite support included in $A_1 \times A_2 \times A_3 = [0, a_1] \times [0, a_2]\, x\, [0, a_3]$. F_i stands for the marginal cdf defined on $\mathbb{R}_+$, $i = 1, 2, 3$. By the Ji$\widetilde{r}$ina theorem, there exists a conditional cdf of X_1 with respect to X_2 and X_3 denoted F_1^{23} and a conditional cdf F_2^3 of X_2 with respect to X_3 such that for any $(x_1, x_2, x_3) \in A_1 \times A_2 \times A_3$

$$F(x_1, x_2, x_3) = \int\limits_{[0,x_2]\times[0,x_3]} F_1^{23}(x_1|X_2 = r, X_3 = t) dF_2^3(r|X_3 = t) dF_3(t). \tag{19}$$

The welfare associated to F is given by

$$W_F := \int\limits_{A_1 \times A_2 x A_3} U(x_1, x_2, x_3) dF(x_1, x_2, x_3)$$

and the welfare variation between the two situations is now defined as

$$\triangle W_U := W_F - W_{F^*} := \int\limits_{A_1 \times A_2 x A_3} U(x_1, x_{2,} x_3) \triangle dF(x_1, x_2, x_3).$$

In performing the integration and in presenting the results, it is convenient to define[5]

$H_i(x_i) = \int\limits_0^{x_i} F_i(r) dr, \qquad i = 1, 2, 3$

$H_i(x_i; x_j, x_k) = \int\limits_0^{x_i} F(r, x_{j,} x_k) dr$ and $H_i(x_i; x_j, a_k) = H_i(x_i; x_j)$ for any i, j, k.

X_3 always plays the role of a compensated variable, while X_1 is always a compensating variable. As for X_2, we consider two cases. In the first one, X_2 plays both roles. In the second one, it is only a compensated variable.

In the first configuration examined, the first variable can compensate for both deficiencies in the two other variables. Besides, the second variable can be used to compensate for a low level in the third variable as well. This configuration is labelled the *full compensation* situation. As an illustration in the spirit of our second illustration given in Section 2, suppose that the social planner has to evaluate the welfare of dynasties of three generations. X_1 stands for the life cycle income (or its rank)of of the current dynastie while X_2 (respectively X_3) stands for the life cycle income of the fathers' (resp. grandfathers') dynasty. Income of the past generations are viewed as handicaps for the realization of the income of the present generation.

$$\begin{aligned} \text{Let } \mathcal{U}^6 &= \{U_1, U_2, U_3 \geq 0, U_{11} \leq 0, U_{22} \leq 0, U_{33} \leq 0, \\ U_{12} &\leq 0,\ U_{13} \leq 0, U_{23} \leq 0,\ U_{121} \geq 0, U_{131} \geq 0, U_{232} \geq 0, U_{123} \geq 0, U_{1123} = 0\} \end{aligned}$$

Except $U_{123} \geq 0$ and $U_{1123} = 0$, the other signs are just a natural extension to the three goods case of the assumptions made in the two goods case. The former restriction must be put together with the assumption that $U_{12} \leq 0$ to be interpreted accuretly. For instance, in the dynasty story, we already now that marginal utility of the son's income is decreasing with the level of the father's income. Now we impose that it decreases with a decreasing rate relative to the level of the grandfather's income. Since the handicaps are just the opposite of the attributes, the compensation is all the more required that the handicap is strong, i.e, that the level of the grandfather's income is low. The latter assumption means that the decline of the social marginal utility of income is additively separable in attributes 2 and 3.

[5] In the proofs, the letter H always means that there exists a variable to which F has been integrated once. This (these) variable(s) appears as a subscript. The semi-colon indicates that the variable at the right of the semi-colon has been "integrated" once less than the variable at the left. A comma between two variables reveals that they have been "integrated" the same number of times.

Proposition 3.1 *The Full Compensation Theorem. Let F and F^* be two CdFs.*

$$F D_{\mathcal{U}^6}\, F^* \qquad (\mathrm{A}_5)$$
$$\Updownarrow$$
$$\triangle H_3(x_3) \le 0\,, \quad \forall x_3 \in X_3, \qquad (\mathrm{B}_5)$$
$$\Delta H_1(x_1; x_2) \le 0, \forall x_i \in X_i,\ i = 1, 2 \qquad (\mathrm{C}_5)$$
$$\Delta H_1(x_1; x_3) \le 0, \forall x_i \in X_i,\ i = 1, 3 \qquad (\mathrm{D}_5)$$
$$\Delta H_1(a_1; x_2, x_3) \le 0, \forall x_i \in X_i,\ i = 2, 3 \qquad (\mathrm{E}_5)$$
$$\Delta H_2(x_2;\, x_3) \le 0, \forall x_i \in X_i,\ i = 2, 3 \qquad (\mathrm{F}_5)$$

Proposition 2 allows to find the counterpart of the second degree stochastic dominance conditions in terms of Lorenz curves. Conditon B_5 requires that the GL curve of the compensating variable x_3 is above for the dominant distribution. The other conditions requires to define the associated RGL curves. For any $i, j, k = 1, 2, 3$, for each x_j in X_j, the function $F^i_{x_j}$ is defined on X_i by the equation

$$F^i_{x_j}(x_i) = F(x_i, x_j, a_k).$$

For a given x_j, $F^i_{x_j}(x_i)$ is at most equal to $F(a_i, x_j, a_k) = F_j(x_j)$. The right inverse is given by

$$\forall p \in [0, F_j(x_j)],\ F^{i-1}_{x_j}(p) = \sup_{F^i_{x_j}(x_i) \le p} x_i.$$

Let define the RGL curve for the joint distribution CDF on $[0, F_2(x_2)]$ by

$$\mathcal{C}_{F^i_{x_j}}(p) = \int_0^p F^{i-1}_{x_j}(t)dt \qquad (20)$$

Finally $F_{x_2,x_3}(x_1) = F(x_1, x_2, x_3)$

Corollary 3.2 *If $F D_{\mathcal{U}^5}\, F^*$, then*

$$\mathcal{C}_{F^1_{x_3}}(p) \ge \mathcal{L}_{F^*_{x_3}}(p)\,, \quad \forall p \in [0, 1], \qquad (\mathrm{B}_{L_3})$$
$$\&\ \mathcal{C}_{F^1_{x_2}}(p) \ge \mathcal{C}_{F^{*1}_{x_2}}(p), \quad \forall p \in [0, \min(F_2(x_2), F^*_2(x_2)], \forall x_2 \in X_2, \qquad (\mathrm{C}_{L_{12}})$$
$$\&\ \mathcal{C}_{F^1_{x_3}}(p) \ge \mathcal{C}_{F^{*1}_{x_3}}(p), \quad \forall p \in [0, \min(F_3(x_3), F^*_2(x_3)], \forall x_3 \in X_3, \qquad (\mathrm{D}_{L_{13}})$$
$$\&\ \mathcal{C}_{F^2_{x_3}}(p) \ge \mathcal{C}_{F^{*2}_{x_3}}(p), \quad \forall p \in [0, \min(F_3(x_3), F^*_2(x_3)], \forall x_3 \in X_3, \qquad (\mathrm{D}_{L_{23}})$$
$$\&\ \int_0^{a_1} x_1 \Delta dF_{x_2,x_3}(x_1)dx_1 \ge 0, \qquad \forall x_i \in X_i,\ i = 2, 3. \qquad (\mathrm{E}_L)$$

Proof. To derive condition E_L, it suffices to integrate by part condition D_5. ■

These necessary conditions are sufficient when the marginal distributions of X_2 and X_3 for F dominate stochasticly at order 1 their counterparts for F^*. Conditions $\mathrm{C}_{L_{12}}$,$\mathrm{D}_{L_{13}}$,$\mathrm{D}_{L_{23}}$ are easy to remenber. The SRGL test has to be performed two by two. It is convenient that there is only one compensated attributed used to defined the groups according to which one has to check dominance. Mind you, this is not true for the last condition. Indeed, a subgroup is described by a maximal value for each compensated variable. For each subgroup so defined, the average income must be larger for the dominant distribution than for the dominated disitribution. It can be noticed that this condition vanishes, when $U_{123} = 0$, namely when the marginal utility function of the compensating good is separable in the two compensated goods.

Introducing more separability assumptions helps us to specify some interesting particular cases of the broad one examined above. Suppose that the first attribute compensates the second one, the second compensating the third - there is a *chain compensation*. More precisely, we are interested in finding conditions ensuring dominance for the following class.

$$\begin{aligned} \text{Let } \mathcal{U}^7 &= \{U_1, U_2, U_3 \geq 0, U_{11} \leq 0, U_{22} \leq 0, U_{33} \leq 0, \\ U_{12} &\leq 0, U_{23} \leq 0, U_{13} = 0,\ U_{121} \geq 0, U_{232} \geq 0\} \end{aligned}$$

The following example provides an illustration of the restrictions imposed on the signs of the social marginal valuations. We intend to design a social welfare function which generates daily choices made by hospitals to allocate resources among ill persons. Assume that the first attribute is income, the second attribute is qalys, the third attribute is age. Let us remind the reader that qalys (quality-adjusted life years) are computed as extra years of life a given treatment gives people, adjusted for quality; better years count more that worse ones. The traditional assumptions of increasingness and concavity with respect to each variable separately are not likely open to debate. For instance, regarding the age variable, it is common sense to say that an additonal year of life provides extra pleasure at any age, but it is likely more pronounced when you are young. In the following reasoning, it is important to remind that a young is just a person "poor in length of life". Consider now a young and and old person for which the qalys associated to some similar treatment are identical. It turns out that the priority is generaly given to the young in that circumstance, see for instance Barrett (2002), which translates in imposing the negativity of U_{23}. I might be the case also that the poor person will not have anything to pay at hospital, even not the board, as for instance in France with the CMU (Universal medical covering), a case figured out by the negativity of U_{12}. It is difficult to find examples of tax or transfer system which discriminate according to age. The age factor, per se, does not seem to represent a relevant characteristic for redistribution. The assumption of nullity of the U_{13}, which implies an additive separability of the utility function with respect to income and age, captures this idea. Therefore, income can compensate for a low qalys, qalys can compensate for age meaning that young get some priority, but income is not used to compensate young to be young. We supplement these assumptions by adding that compensation is all the more necessary that people are poor in the compensated variables. The priority for a medical treatment will become even more obvious if one of the ill person is just a child ($U_{232} \geq 0$), while the choice between two old persons will become less transparent on this basis. A full coverage of the medical treatment is all the more required than the person is poor ($U_{121} \geq 0$).

Proposition 3.3 *The Chain Compensation Theorem. Let F and F*be two CdFs..*

$$F D_{\mathcal{U}^7} F^* \tag{A$_6$}$$

$$\Updownarrow$$

$$\triangle H_3(x_3) \leq 0\,, \quad \forall x_3 \in X_3 \tag{B$_5$}$$

$$\Delta H_1(x_1; x_2) \leq 0\,, \forall x_i \in X_i\ ,\ \ i = 1, 2 \tag{C$_5$}$$

$$\Delta H_2(x_2; x_3) \leq 0, \forall x_i \in X_i\ ,\ \ i = 2, 3 \tag{F$_5$}$$

As an application of corollary 4.1, conditions B_{L_3},$C_{L_{12}}$,$D_{L_{13}}$,$D_{L_{23}}$ are necessary to check dominance according to class $\mathcal{U}^7$. They are sufficient when the marginal distribution of the third attribute dominates stochastically to the first order. Equivalently we can state a companion corollary to corollary 2.1, when we drop condition $U_{33} \leq 0$ from the list defining $\mathcal{U}^7$ and we label $\mathcal{U}^{71}$ the corresponding family. We state

Corollary 3.4 *Let F and F^* be two CdFs..*

$$F D_{\mathcal{U}^7} F^* \tag{A$_{61}$}$$

$$\Updownarrow$$

$$\triangle F_3(x_3) \leq 0\,, \quad \forall x_3 \in X_3 \tag{B$_{61}$}$$

$$\mathcal{C}_{F^1_{x_3}}(p) \geq \mathcal{C}_{F^{*1}_{x_3}}(p), \quad \forall p \in [0, \min(F_3(x_3)], \forall x_3 \in X_3 \tag{C$_{61}$}$$

$$\mathcal{C}_{F^2_{x_3}}(p) \geq \mathcal{C}_{F^{*2}_{x_3}}(p), \quad \forall p \in [0, \min(F_3(x_3)], \forall x_3 \in X_3. \tag{D$_{61}$}$$

The last configuration called the *single compensation* configuration occurs when the first attribute can compensate for the two others attributes. For instance, income may compensate for health and family size.

$$\begin{aligned} \text{Let } \mathcal{U}_8 &= \{U_1, U_2, U_3 \geq 0, U_{11} < 0, U_{22} < 0, U_{33} < 0, \\ U_{12} &\leq 0, U_{13} \leq 0, U_{23} = 0,\ U_{121} > 0, U_{131} > 0\} \end{aligned}$$

The assumption of the nullity of U_{23} means that the utility function is additely separable in the second and third attributes, namely,

$$U(x_1, x_2, x_3) = \varphi(x_1, x_2) + \psi(x_1, x_3) \tag{21}$$

Proposition 3.5 *The Single Compensation Theorem. Let F and F^* two cdfs.*

$$F D_{\mathcal{U}^8}\ F^* \tag{A_7}$$

$$\Updownarrow$$

$$\begin{aligned} \triangle H_i(x_i) &\leq 0\,, \quad \forall x_i \in X_i \quad i = 2, 3 && (B_5) \\ \Delta H_1(x_1; x_2) &\leq 0, \forall x_i \in X_i,\ i = 1, 2 && (C_5) \\ \Delta H_1(x_1; x_3) &\leq 0, \forall x_i \in X_i,\ i = 1, 3. && (22) \end{aligned}$$

If we dispose of conditions $U_{22} \leq 0$ and $U_{33} \leq 0$ from the list defining $\mathcal{U}^8$ and we label $\mathcal{U}^{81}$ the corresponding family, we obtain

Corollary 3.6 *Let F and F^* be two CdFs.*

$$F D_{\mathcal{U}^{81}} F^* \tag{A$_{71}$}$$

$$\Updownarrow$$

$$\triangle F_2(x_2) \leq 0\,, \quad \forall x_2 \in X_2, \&\ \ \triangle F_3(x_3) \leq 0\,, \quad \forall x_3 \in X_3 \tag{B$_{71}$}$$

$$\mathcal{C}_{F^1_{x_2}}(p) \geq \mathcal{C}_{F^{*1}_{x_2}}(p), \quad \forall p \in [0, \min(F_2(x_2)], \forall x_2 \in X_2 \tag{C$_{71}$}$$

$$\mathcal{C}_{F^2_{x_3}}(p) \geq \mathcal{C}_{F^{*2}_{x_3}}(p), \quad \forall p \in [0, \min(F_3(x_3)], \forall x_3 \in X_3. \tag{D$_{71}$}$$

The message which follows from studying the three-goods case is the following. Extending the two-goods result to a more general setting is possible, but introducing some restrictive separability assumptions is the price to pay for obtaining palatable results.

4 Concluding comments

Our results provide some integration of the need approach and of the multidimensional one. In applied studies, we have three methods at our disposal, the needs approach with fixed marginal distributions of needs, the needs approch with variable marginal distributions of needs, and the truly multidimensional one exposed here. In most and probably most cases the first attribute is income and in the following we keep this in mind. If we are asking an instructor manual, we likely suggest to limit the use of the former approach to an assessment of the sole fiscal policy; the needs distribution may be influenced by other public policies and keeping invariant the needs distribution helps to discard these other policies from debate. At the opposite the multivariate approach seems well suited when we are keen to assess the global impact of all public policies. In the middle, it may be the case that we would like to appraise the global impact of public policy but we are uncomfortable with the sign of the social marginal valuation of some need. The example of family size provides an illustration. Is a child a social cost or a social good ? We brought some answers in section 2 which support the view that a child may be viewed as as a cost, keeping the household budget constant. But we admit that in that matter, there is some room for differences in opinions. In such a case, choosing the needs approch with variable marginal distributions of needs may be right choice.

References

[1] Arneson R. , 1989, "Equality and equal opportunity for welfare," *Philosophical Studies 56* , 77-93

[2] Atkinson A. B. , 1970, "On the Measurement of Inequality," *J. Econ. Theory 2* , 244-263.

[3] Atkinson A. B. , 1981, "The measurement of income mobility" in P.J. Eggelshoven and L.J. van Gemerden (eds), *Inkommens Verdeling en Openbard Financien.* Leiden: Het Spectrum.

[4] Atkinson A. B., F. Bourguignon, 1982, "The Comparison of Multi-Dimensioned Distributions of Economic Status," *Review of Economic Studies* 49: 181-201.

[5] Atkinson A. B. and F. Bourguignon, 1987, Income Distribution and Differences in Needs, in "*Arrow and the foundation of the theory of economic policy,*" (G.R. Feiwel, ed.), Macmillan, London.

[6] Atkinson A. B. and F. Bourguignon, 2000, Income Distribution and Economics, in "*Handbook of Income Distribution*" (A.B. Atkinson and F. Bourguignon, ed.), North-Holland.

[7] Barrett R.,2002, Qualys, Rights and the optimal provision of health services, Presented to the Sixth Social Choice and Welfare Congress in Pasadena.

[8] Bazen S. and P. Moyes 2001, "International Comparisons of Income Distributions " Forthcoming in *Research in Economic Inequality.*

[9] Bourguignon F., 1989, "Family Size and Social Utility. Income distribution dominance criteria," *J. Econometrics* 42, 67-80.

[10] BILLINGSLEY P. .,1986 "Probability and Measure", Second Edition, Wiley-Interscience.

[11] CHAMBAZ C. AND E. MAURIN, 1998, "Atkinson and Bourguignon's Dominance Criteria: Extended and Applied to the Measurement of Poverty in France," *Rev. Income Wealth* 44, 497-513.

[12] COHEN G., 1989," On the currency of egalitarian justice" *Ethics* ,99, 1981, 906-944.

[13] DWORKIN R., 1981," What is equality ? Part 2: Equality of Resources," *Philosophy and Public Affairs* ,10, 283-245.

[14] EBERT U., 1995, "Income Inequality and Differences in Household Size," *Math. Soc. Sci.* 30, 37-55.

[15] EBERT U., 1997, "Social Welfare when Needs Differ: An Axiomatic Approach," *Economica* 64 , 233-344.

[16] EBERT U., 1999, "Using Equivalence Income of Equivalent Adults to Rank Income Distributions," *Soc. Choice Welfare* 16, 233-258.

[17] EBERT U., 2000, "Sequential Generalized Lorenz Dominance and Transfer Principles," *Bulletin Econ. Research* 52, 113-123.

[18] FISHBURN P.C AND R.G VICKSON., 1978, "Theoretical foundations of stochastic dominance" in Whitmore G. A. et Findlay M. C (eds) *Stochastic Dominance*, Lexington Books.

[19] FLEURBAEY, M., C. HAGNERE AND A. TRANNOY, 2001, "Welfare analysis when equivalence scales differ," Mimeo THEMA forthcoming in *Journal of Economic Theory.*

[20] FLEURBAEY, M., C. HAGNERE AND A. TRANNOY, 2001, "Minima sociaux et cumul d'emploi : évaluation d'une réforme à l'aide d'un nouveau critère de dominance" *Revue Economique,* 53, 6, 1205-1234.

[21] FLEURBAEY M., 1998, "Equality among responsible individuals" in *Freedom in Economics, New Perspective in Normative Analysis*, ed. by J.F. Laslier, M. Fleurbaey, N.Gravel and A.Trannoy, Routledge, 206-234.

[22] FOSTER J. E. AND A. F. SHORROCKS, 1988, "Poverty Orderings and Welfare Dominance," *Soc. Choice Welfare* 5, 179-198.

[23] GENET J, 1976, "*Mesure et Intégration*", Vuibert, Paris.

[24] JENKINS S. P. AND P. J. LAMBERT, 1993, "Ranking Income Distributions when Needs Differ," *Rev. Income Wealth* 39, 337-356.

[25] HUANG, C. C., D. KIRA, AND I. VERTINSKY, 1978, "Stochastic Dominance Rules for Multi-Attribute Utility Functions", *The Review of Economic Studies*, vol. 45, Issue 3, 611-615.

[26] KOLM S. C., 1968, "The Optimal Production of Social justice" in H. Guitton, J. Margolis (eds.), *Economie Publique,* Paris, CNRS.

[27] Kolm S. C., 1977, "Multidimensional Egalitarianisms," *Quarterly Journal Of Economics*, 91:1-13.

[28] Koshevoy G., 1995, "Multivariate Lorenz Majorizations," *Social Choice and Welfare* 12: 93-102.

[29] Koshevoy G., 1998, "The Lorenz Zonotope and Multivariate Majorizations," *Social Choice and Welfare* 15: 1-14.

[30] Koshevoy G., K. Mosler, 1996, "The Lorenz Zonoïd of a Multivariate Distributions", *Journal of the American Statistical Association* 91(434): 873-882.

[31] Lambert P.J. and X Ramos, 2000, "Welfare Comparisons: Sequential Procedures for Homogeneous Populations," forthcoming in *Economica.*

[32] Le breton M., 1986, "Essais sur les Fondements de l'Analyse Economique de l'Inégalité," Thèse pour le Doctorat d'Etat, Université de Rennes I.

[33] Marshall A., I. Olkin, 1979, *Inequalities : Theory of Majorization and Its Applications,* Academic Press, New York.

[34] Metivier M., 1982, *Notions Fondamentales de la Théorie des Probabilités,* Dunod, Paris (*In French).*

[35] P. Moyes, 1999, Comparisons of heterogeneous distributions and dominance criteria, *Economie et Prévision* 138-139, 125-146 (in French).

[36] Ok E. A. and P. J. Lambert, 1999, "An Evaluating Social Welfare by Sequential Generalized Lorenz Dominance," *Econ. Letters* 63, 39-44.

[37] Roemer J., 1985, "Equality of talent"*Economics and Philosophy*,1, 151-187.

[38] Roemer J., 1993, "A pragmatic theory of responsability for the egalitarian planner" *Philosophy and Public Affairs 22, 146-166.*

[39] Roemer J., 1996, *Theories of Distributive Justice,* Harvard University Press, Cambridge, Mass.

[40] Sen A. K., 1973, *On Economic Inequality*, Clarendon Press, Oxford.

[41] Sen A. K., 1985, *Commodities and Capabilities,* North-Holland, Amsterdam.

[42] Sen A. K., 1987, *The standard of Living,* Cambridge University Press, Cambridge.

[43] Sen A. K., 1992, *Inequality Re-examined,* Clarendon Press, Oxford.

[44] Schokkaert E. and K. Devooght.,1998, "The empirical acceptance of compensation axioms" in *Freedom in Economics, New Perspective in Normative Analysis*, ed. by J.F. Laslier, M. Fleurbaey, N.Gravel and A.Trannoy, Routledge, 257-281.

[45] Shorrocks A. F., 1983, "Ranking Income Distributions," *Economica* 50, 1-17.

[46] Shorrocks A. F., 1995,"Inequality and Welfare Evaluation of Heterogeneous Income Distributions," Discussion Paper No. 447, Department of Economics, University of Essex,

[47] Shorrocks A. F. and Foster J.E., 1987, “Transfer Sensitive Inequality Measures,” *Review of Economic Studies* 54, 485-497.

APPENDIX

Proof of Proposition 2.1:

*Sufficiency.*Using the extension of Fubini's theorem to conditional probability measures (see Billingsley (1986) exercice 18.25 p.247) we write,

$$W_F = \int_{A_2} \left[\int_{A_1} U(x_1, x_2) dF_1^2(x_1 | X_2 = x_2) \right] dF_2(dx_2). \tag{23}$$

Consider the inner integral. $U(x_1, x_2)$ is a differentiable function positive in A_1 while F_1^2 is an increasing right continuous function in A_1. Then they have no common point of discontinuity in A_1. Then the classical formulae of integration by parts applies (see for instance Billingsley 1986 theorem 18.4 p.240). It gives

$$\int_{A_1} U(x_1, x_2) dF_1^2(x_1 | X_2 = x_2) = U(a_1, x_2) F_1^2(a_1 | X_2 = x_2) - U(0, x_2) F_1^2(0 | X_2 = x_2) \tag{24}$$

$$- \int_{(0,a_1]} U_1(x_1, x_2) dF_1^2(x_1 | X_2 = x_2)$$

or

$$\int_{1} U(x_1, x_2) dF_1^2(x_1 | X_2 = x_2) = U(a_1, x_2) F_1^2(a_1 | X_2 = x_2)$$

$$- \int_{A_1} U_1(x_1, x_2) dF_1^2(x_1 | X_2 = x_2)$$

Integrating the second term of the RHS of the above expression by part once again and substituting in (23), we get

$$W_F = \int_{A_2} U(a_1, x_2) F_1^2(a_1 | X_2 = x_2) dF_2(x_2) \tag{25a}$$

$$- \int_{A_2} \left[U_1(a_1, x_2) \int_{A_1} F_1^2(x_1 | X_2 = x_2) dx_1 \right] dF_2(x_2) \tag{25b}$$

$$+ \int_{A_2} \left[\int_{A_1} U_{11}(x_1, x_2) \int_{[0,x_1]} F_1^2(s | X_2 = x_2) ds dx_1 \right] dF_2(x_2) \tag{25c}$$

Integrating by part the first term of the RHS of the above expression gives,

$$\int_{A_2} U(a_1,x_2)F_1^2(a_1|X_2=x_2)dF_2(x_2) = U(a_1,a_2)\left[\int_{A_2} F_1^2(a_1|X_2=x_2)dF_2(x_2)\right]$$
$$-\int_{A_2} U_2(a_1,x_2)\left[\int_{[0,x_2]} F_1^2(a_1|X_{2=}t)dF_2(t)\right]dx_2$$

Evaluating by using the definition of a conditional probability distribution (1),it reduces to

$$\int_{A_2} U(a_1,x_2)F_1^2(a_1|X_2=x_2)dF_2(x_2) = U(a_1,a_2)F(a_1,a_2) \tag{26}$$
$$-\int_{A_2} U_2(a_1,x_2)F(a_1,x_2)dx_2 \tag{27}$$

Integrating by part the second term of the RHS of (25b) with respect to x_2 gives

$$-\int_{A_2}\left[U_1(a_1,x_2)\int_{A_1} F_1^2(x_1|X_2=x_2)dx_1\right]dF_2(x_2) =$$
$$-U_1(a_1,a_2)\left[\int_{A_2}\left[\int_{A_1} F_1^2(x_1|X_2=x_2)dx_1\right]dF_2(x_2)\right]$$
$$+\int_{A_2} U_{12}(a_1,x_2)\left[\int_{[0,x_2]}\left[\int_{A_1} F_1^2(x_1|X_{2=}t)dx_1\right]dF_2(x_2)\right]dx_2$$

By Fubini,

$$\left[\int_{[0,x_2]}\left[\int_{A_1} F_1^2(x_1|X_{2=}t)dx_1\right]dF_2(x_2)\right] = \int_{A_1}\left[\int_{[0,x_2]} F_1^2(x_1|X_{2=}t)dF_2(x_2)\right]dx_1 = \int_{A_1} F(x_1,x_2)dx_1$$

it reduces to

$$-\int_{A_2}\left[U_1(a_1,x_2)\int_{A_1} F_1^2(x_1|X_2=x_2)dx_1\right]dF_2(x_2) = -U_1(a_1,a_2)\int_{A_1} F_1(x_1)dx_1$$
$$+\int_{A_2} U_{12}(a_1,x_2)\left[\int_{A_1} F(x_1,x_2)dx_1\right]dx_2$$

Similarly, integrating by part the third term of (25c) with respect to x_2, we obtain

$$\int_{A_2}\left[\int_{A_1} U_{11}(x_1,x_2)\int_{[0,x_1]} F_1^2(s|X_2=x_2)ds\right]dF_2(x_2) =$$
$$\int_{A_1} U_{11}(x_1,a_2)\left[\int_{A_2}\left[\int_{[0,x_1]} F_1^2(s|X_2=x_2)ds\right]dF_2(x_2)\right]$$

$$-\int_{A_2}\int_{A_1} U_{112}(x_1,x_2)\left[\int_{[0,x_2]}\left[\int_{[0,x_1]} F_1^2(s|X_{2=}s)ds\right]dF_2(x_2)\right]dx_1dx_2$$

which reduces to

$$= \int_{A_1} U_{11}(x_1,a_2)\left[\int_{[0,x_1]} F_1(s)ds\right]dx_1 \tag{28}$$
$$-\int_{A_2}\int_{A_1} U_{112}(x_1,x_2)\left[\int_{[0,x_1]} F(s,x_2)ds\right]dx_1dx_2$$

Recapitulating, it yields

$$\begin{aligned}
W_F &= U(a_1,a_2)F(a_1,a_2)\\
&-\int_{A_2} U_2(a_1,x_2)F_2(x_2)dx_2\\
&-U_1(a_1,a_2)\int_{A_1} F_1(x_1)dx_1\\
&+\int_{A_2} U_{12}(a_1,x_2)\left[\int_{A_1} F(x_1,x_2)dx_1\right]dx_2\\
&+\int_{A_1} U_{11}(x_1,a_2)\left[\int_{[0,x_1]} F_1(s)ds\right]dx_1\\
&-\int_{A_2}\int_{A_1} U_{112}(x_1,x_2)\left[\int_{[0,x_1]} F(s,x_2)ds\right]dx_1dx_2
\end{aligned}$$

Therefore

$$\Delta W_U = \int\limits_{A_1 \times A_2} U(x_1,x_2)\triangle dF(x_1,x_2)) = U(a_1,a_2)\Delta F(a_1,a_2) \tag{29a}$$

$$-\int_0^{a_2} U_2(a_1,x_2)\Delta F_2(x_2)dx_2 \tag{29b}$$

$$-U_1(a_1,a_2)\int_0^{a_1} \Delta F_1(x_1)dx_1 \tag{29c}$$

$$+\int_0^{a_2} U_{12}(a_1,x_2)\left[\int_0^{a_1} \Delta F(x_1,x_2)dx_1\right]dx_2 \tag{29d}$$

$$+\int_0^{a_1} U_{11}(x_1,a_2)\left[\int_0^{x_1} \Delta F(s,a_2)ds\right]dx_1 \tag{29e}$$

$$-\int_0^{a_1}\int_0^{a_2} U_{112}(x_1,x_2)\left[\int_0^{x_1} \Delta F(s,x_2)ds\right]dx_1 dx_2\,. \tag{29f}$$

It follows that integrating by part the second term in the RHS term and evaluating some other terms yields

$$\Delta W_U = U(a_1,a_2)\Delta F(a_1,a_2) - U_2(a_1,a_2)\Delta H_2(a_2) \tag{30}$$

$$+\int_0^{a_2} U_{22}(a_1,x_2)\Delta H_2(x_2)dx_2 \tag{31}$$

$$-U_1(a_1,a_2)\Delta H_1(a_1) + \int_0^{a_2} U_{12}(a_1,x_2)\Delta H_1(a_1;x_2)dx_2 \tag{32}$$

$$+\int_0^{a_1} U_{11}(x_1,a_2)\Delta H_1(x_1)dx_1 \tag{33}$$

$$-\int_0^{a_1}\int_0^{a_2} U_{112}(x_1,x_2)\Delta H_1(x_1;x_2)\, dx_1 dx_2. \tag{34}$$

The first term vanishes and, since Condition C implies $\Delta H_1(x_1) \leq 0, \forall (x_1) \in X_1$, the conclusion follows.

Necessity. The proof follows some standard argument (Fishburn and Vickson, 1978, p76). Let us suppose that condition B is not fullfilled. Then there exists $x_2^* \in X_2$ such that $\Delta H_2(x_2^*) > 0$.

Let us define the utility function V_B as follows.

$$\forall\ x_1 \in X_1, V_B(x_1,x_2) = x_2 - x_2^*,\ \forall\ x_2 \leq x_2^* \text{ and } V_B(x_1,x_2) = 0,\ \forall\ x_1 > x_1^*.$$

It is readily shown that $\triangle W_V = -\Delta H_2(x_2^*)$. Clearly V_B does not belong to $\mathcal{U}^2$, but it is possible to find an approximation of V_B which belongs to $\mathcal{U}^2$.

Let us supposed that condition C is not fullfilled. Then there exists $(x_1^*, x_2^*\ ) \in X_1$ x X_2 such that $\Delta H_1(x_1^*; x_2^*) > 0$.

Let us define the utility function V_C such that

$$\forall\, x_2 \leq x_2^*, \;\; \forall\, x_1 \leq x_1^* \qquad\qquad V_C(x_1, x_2) = x_1 - x_1^*, \;. \tag{35}$$
$$\text{otherwise} \;\; V_C(x_1, x_2) = 0, \tag{36}$$

It is readily shown that $\triangle W_{V_C} = -\Delta H_1(x_1^*; x_2^*)$. Clearly V_C does not belong to $\mathcal{U}^2$ but it is possible to find an approximation of V_C which belongs to $\mathcal{U}^2$.

Proof of proposition 2.3

i) Proof of A_L [6].

Necessity. We can use Young inequality (see e.g. Genet (1976) theorem 1 p. 195) since either $F_2(x_2) = 0$ or $F_2^{-1}(x_2) = 0$. Iindeed it degenerates in our case in an equality,

$$\int_0^{x_2} F_2(s)ds = x_2 F_2(x_2) - \int_0^{F_2(x_2)} F_2^{-1}(t)dt$$

which may be rewritten with $q = F_2(x_2)$

$$H_2(x_2) = \int_0^{x_2} F_2(s)ds = qF_2^{-1}(q) - \int_0^{q} F_2^{-1}(t)dt$$

and with a similar expression for $H_2^*(x_2)$ with $q^* = F_2^*(x_2)$ one yields

$$\triangle H_2(x_2) = qF_2^{-1}(q) - q^* F_2^{*-1}(q^*) - [\int_0^{q} F_2^{-1}(t)dt - \int_0^{q^*} F_2^{*-1}(t)dt] \tag{37}$$

or

$$\triangle H_2(x_2) = qF_2^{-1}(q) - q^* F_2^{*-1}(q^*) - sgn(q - q^*) \int_{q^*}^{q} F_2^{-1}(t)dt - [\mathcal{L}_{F_2}(q^*) - \mathcal{L}_{F_2^*}(q^*)\,]$$

Using $F_2^{-1}(q) = F_2^{*-1}(q^*)$

$$\triangle H_2(x_2) = \left[F_2^{-1}(q)(q - q^*) - sgn(q - q^*) \int_{q^*}^{q} F_2^{-1}(t)dt \right] - [\mathcal{L}_{F_2}(q^*) - \mathcal{L}_{F_2^*}(q^*)\,] \tag{38}$$

Applying the mean-value theorem for integrals, the term in brackets is always positive, and therefore we deduce that $\triangle H_2(x_2) \leq 0 \Rightarrow \mathcal{L}_{F_2}(F_2^*(x_2)) - \mathcal{L}_{F_2^*}(F_2^*(x_2)) \geq 0$. Since $F_2^*(x_2) \in [0, 1]$, $\triangle H_2(x_2) \leq 0$, $\forall x_2 \in X_2$ implies $\mathcal{L}_{F_2}(p) \geq \mathcal{L}_{F_2^*}(p)$, $\forall p \in [0, 1]$.

Sufficiency. Suppose that there exists $q \in [0, 1]$ such that $\mathcal{L}_{F_2}(q) \geq \mathcal{L}_{F_2^*}(q)$. Then, $\exists$ $x_2 \in X_2 | x_2 = F_2^{-1}(q)$ and $\exists\, q^* \in [0, 1]$ with $q = F_2^*(x_2)$ such that equation (37) is valid. This equation can be expressed as well as

$$\triangle H_2(x_2) = \left[sgn(q^* - q) \int_{q}^{q^*} F_2^{*-1}(t)dt - (q^* - q)F_2^{*-1}(q^*) \right] - [\mathcal{L}_{F_2}(q) - \mathcal{L}_{F_2^*}(q)\,]$$

[6] Shorrocks (1983) gives a proof of step 1 for a discrete probability based on the Hardy-Littlewood-Polya theorem. Le Breton (1986) p.88-89 gives a proof for a general probability distribution based on Young inequality. Our proof is adapted from his.

Applying the mean-value theorem for integrals, the term in brackets is always negative and $\mathcal{L}_{F_2}(q) \geq \mathcal{L}_{F_2^*}(q). \geq 0$ implies $\triangle H_1(F_2^{-1}(q) \leq 0$.

Since the range of $F_2^{-1}(p)$ is X_2, $\mathcal{L}_{F_2}(p) \geq \mathcal{L}_{F_2^*}(p)$, $\forall p \in [0,1]$ implies $\triangle H_2(x_2) \leq 0$, $\forall x_2 \in X_2$.

ii) Proof of B_L.

We consider two cases. In both cases, we choose a fixed x_2 in X_2.

Case 1. $x_2 \in X_2$ is such that $F_2^*(x_2) \leq F_2(x_2)$. We prove that $\triangle H_1(x_1; x_2) \leq 0$, $\forall x_1 \in X_1 \Rightarrow \mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F_{x_2}^*}(p)$, $\forall p \in [0, F_2^*(x_2)]$.

Starting with the definition of $H_1(x_1; x_2)$ (7) and using Young inequality, one get

$$\int_0^{x_1} F_{x_2}(s)ds = x_1 F_{x_2}(x_1) - \int_0^{F_{x_2}(x_1)} F_{x_2}^{-1}(t)dt \tag{39}$$

or with $q = F_{x_2}(x_1) \in [0, F_2(x_2)]$

$$H_1(x_1; x_2) = F_{x_2}^{-1}(q)q - \int_0^q F_{x_2}^{-1}(t)dt$$

and with a similar expression for $H_1^*(x_1; x_2)$ with $q^* = F_{x_2}^*(x_1) \in [0, F_2^*(x_2)]$ one yields

$$\triangle H_1(x_1; x_2) = qF_{x_2}^{-1}(q) - q^* F_{x_2}^{*-1}(q^*) - [\int_0^q F_{x_2}^{-1}(t)dt - \int_0^{q^*} F_{x_2}^{*-1}(t)dt] \tag{40}$$

Since $F_2^*(x_2) \leq F_2(x_2)$,we have $F_{x_2}^*(x_1) \leq F_2(x_2)$, and the proof of (i) can be adapted

$$\triangle H_1(x_1; x_2) = qF_{x_2}^{-1}(q) - q^* F_{x_2}^{*-1}(q^*) - sgn(q - q^*) \int_{q^*}^q F_{x_2}^{-1}(t)dt - [\mathcal{C}_{F_{x_2}}(q^*) - \mathcal{C}_{F_{x_2}^*}(q^*)\,]$$

Using $F_{x_2}^{-1}(q) = F_{x_2}^{*-1}(q^*)$

$$\triangle H_1(x_1; x_2) = \left[F_{x_2}^{-1}(q)(q - q^*) - sgn(q - q^*) \int_{q^*}^q F_{x_2}^{-1}(t)dt \right] - [\mathcal{C}_{F_{x_2}}(q^*) - \mathcal{C}_{F_{x_2}^*}(q^*)\,] \tag{41}$$

Applying the mean-value theorem for integrals, the term in brackets is always positive, and therefore we deduce that $\triangle H_1(x_1; x_2) \leq 0 \Rightarrow \mathcal{C}_{F_{x_2}}(F_{x_2}^*(x_1)) - \mathcal{C}_{F_{x_2}^*}(F_{x_2}^*(x_1)) \geq 0$. Since $F_{x_2}^*(x_1) \in [0, F_2^*(x_2)]$, $\triangle H_1(x_1; x_2) \leq 0$, $\forall x_1 \in X_1$ implies $\mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F_{x_2}^*}(p)$, $\forall p \in [0, F_2^*(x_2)]$.

Case 2. $x_2 \in X_2$ is such that $F_2(x_2) < F_2^*(x_2)$. We prove that $\triangle H_1(x_1; x_2) \leq 0$, $\forall x_1 \in X_1 \Rightarrow \mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F_{x_2}^*}(p)$, $\forall p \in [0, F_2(x_2)]$.

We start with 37 which remains valid. By assumption, $\triangle H_1(x_1; x_2) \leq 0$ for all x_1. Then it is also true for all $x_1 \in X_1$ such that $F_{x_2}^*(x_1) \leq F_2(x_2)$. In that case, the end of the necessity of the proof of case1 remains valid and therefore we deduce that $\triangle H_1(x_1; x_2) \leq 0 \Rightarrow \mathcal{C}_{F_{x_2}}(F_{x_2}^*(x_1)) - \mathcal{C}_{F_{x_2}^*}(F_{x_2}^*(x_1)) \geq 0$. Since $F_{x_2}^*(x_1) \in [0, F_2(x_2)]$, $\triangle H_1(x_1; x_2) \leq 0$, $\forall x_1 \in X_1$ implies $\mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F_{x_2}^*}(p)$, $\forall p \in [0, F_2(x_2)]$. Statement B_L follows.

iii) Proof of C_L. $x_2 \in X_2$ is such that $F_2(x_2) < F_2^*(x_2)$.

In view of B_L, it suffices to prove that $\mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F^*_{x_2}}(p)$, $\forall p \in [0, F_2(x_2)]$ implies $\triangle H_1(x_1; x_2) \leq 0$, $\forall x_1 \in X_1$.

Suppose that there exists $q \in [0, F_2(x_2)]$ such that $\mathcal{C}_{F_{x_2}}(q) \geq \mathcal{C}_{F^*_{x_2}}(q)$. Then, $\exists\, x_1 \in X_1 | x_1 = F^{-1}_{x_2}(q)$ and there exists $q^* = F^*_{x_2}(x_1)$. Starting from Equation (37) which remains valid, and using $F_2(x_2) < F^*_2(x_2)$ which implies that $q \leq F^*_2(x_2)$, one gets

$$\triangle H_1(x_1; x_2) = qF^{-1}_{x_2}(q) - q^* F^{*-1}_{x_2}(q^*) + sgn(q^* - q) \int_q^{q^*} F^{*-1}_{x_2}(t)dt - [\int_0^q F^{-1}_{x_2}(t)dt - \int_0^q F^{*-1}_{x_2}(t)dt]$$

or using $F^{-1}_{x_2}(q) = F^{*-1}_{x_2}(q^*)$

$$\triangle H_1(x_1; x_2) = \left[sgn(q^* - q) \int_q^{q^*} F^{*-1}_{x_2}(t)dt - F^{*-1}_{x_2}(q^*)(q^* - q) \right] - [\mathcal{C}_{F_{x_2}}(q) - \mathcal{C}_{F^*_{x_2}}(q)]$$

Applying the mean-value theorem for integrals, the term in brackets is always negative and $\mathcal{C}_{F_{x_2}}(q) - \mathcal{C}_{F^*_{x_2}}(q) \geq 0$ implies $\triangle H_1(F^{-1}_{x_2}(q); x_2) \leq 0$. Since the range of $F^{-1}_{x_2}(p)$ is X_1, $\mathcal{C}_{F_{x_2}}(p) \geq \mathcal{C}_{F^*_{x_2}}(p)$, $\forall p \in [0, F_2(x_2)]$ implies $\triangle H_1(x_1; x_2) \leq 0$, $\forall x_1 \in X_1$.

Proof of Proposition 2.7 (i)

Sufficiency Starting from the final expression for ΔW_U in the proof of Proposition 1 and integrating by part (33) with respect to x_1 we get

$$\Delta W_U = -U_2(a_1, a_2)\Delta H_2(a_2) \tag{42}$$

$$+ \int_0^{a_2} U_{22}(a_1, x_2)\Delta H_2(x_2)dx_2 \tag{43}$$

$$-U_1(a_1, a_2)\Delta H_1(a_1) + \int_0^{a_2} U_{12}(a_1, x_2)\Delta H_1(a_1; x_2)dx_2 \tag{44}$$

$$+U_{11}(a_1, a_2)\Delta L_1(a_1) - \int_0^{a_1} U_{111}(x_1, a_2)\Delta L_1(x_1)dx_1 \tag{45}$$

$$- \int_0^{a_1} \int_0^{a_2} U_{112}(x_1, x_2)\Delta H_1(x_1; x_2)\, dx_1 dx_2. \tag{46}$$

$\Delta H_1(x_1; a_2) \leq 0 \Rightarrow \Delta L_1(x_1) \leq 0$.The conclusion follows.

Necessity. Similar to proposition 2.1.

Proof of Proposition 2.7(ii)

Sufficiency Starting from the final expression for ΔW_U in the proof of Proposition 1 and integrating by part (31) with respect to x_2 we get

$$\begin{aligned}
\Delta W_U &= -U_2(a_1,a_2)\Delta H_2(a_2) && (47)\\
&\quad +U_{22}(a_1,a_2)\Delta L_2(a_2) - \int_0^{a_2} U_{222}(a_1,x_2)\Delta L_2(x_2)dx_2 && (48)\\
&\quad -U_1(a_1,a_2)\Delta H_1(a_1) + \int_0^{a_2} U_{12}(a_1,x_2)\Delta H_1(a_1;x_2)dx_2 && (49)\\
&\quad +\int_0^{a_1} U_{11}(x_1,a_2)\Delta H_1(x_1)dx_1 && (50)\\
&\quad -\int_0^{a_1}\int_0^{a_2} U_{112}(x_1,x_2)\Delta H_1(x_1;x_2)\, dx_1 dx_2. && (51)
\end{aligned}$$

The conclusion follows.

Necessity. For B_3 and C_3, see proof of proposition 2.1. Now suppose that D_3 is not satisfied. Then there exists $x_2^* \in X_1$ such that $\Delta L_1(x_2^*) > 0$

Let us define the utility function V_{D_3} as follows.

$$\forall\ x_1 \in X_1, V_{D_3}(x_1,x_3) = -1/2(x_3 - x_3^*)^{1/2},\ \forall\ x_3 \le x_3^* \text{ and } V_{D_3}(x_1,x_3) = 0,\ \forall\ x_1 > x_1^*.$$

It is readily shown that $\triangle W_V = -\Delta L_1(x_3^*)$. Clearly V_{D_3} does not belong to $\mathcal{U}^3$, but it is possible to find an approximation of V_{D_3} which belongs to $\mathcal{U}^3$

Proof of Proposition 2.8

Sufficiency Starting from the final expression for ΔW_U in the proof of Proposition 2.1 and integrating by part (32) with respect to x_2 we get

$$\begin{aligned}
\Delta W_U = U(a_1,a_2)\Delta F(a_1,a_2) - U_2(a_1,a_2)\Delta H_2(a_2) && (52)\\
+\int_0^{a_2} U_{22}(a_1,x_2)\Delta H_2(x_2)dx_2 && (53)\\
-U_1(a_1,a_2)\Delta H_1(a_1) + U_{12}(a_1,a_2)\Delta H(a_1,a_2)dx_2 && (54)\\
-\int_0^{a_2} U_{122}(a_1,x_2)\Delta H(a_1,x_2)dx_2 && (55)\\
+\int_0^{a_1} U_{11}(x_1,a_2)\Delta H_1(x_1)dx_1 && (56)\\
-\int_0^{a_1}\int_0^{a_2} U_{112}(x_1,x_2)\Delta H_1(x_1;x_2)\, dx_1 dx_2. && (57)
\end{aligned}$$

Since $\Delta H(a_1,x_2) = \int_0^{x_2} \Delta H_1(a_1;t)dt$, checking $\Delta H_1(x_1;s) \le 0$ for any s ensures that $\Delta H(a_1,x_2) \le 0$ and the conclusion follows.

Necessity. Identical to proposition 2.1

Proof of Proposition 3.1:

Sufficiency. We start from the definition of the welfare function in which the distribution of the first variable is separated by conditioning.

$$W_F = \int\limits_{A_2\times A_3} \left[\int\limits_{A_1} U(x_1,x_2,x_3) dF_1^{23}(x_1|X_2=x_2,X_3=x_3) \right] dF_{23}(x_2,x_3). \tag{58}$$

In the whole proof, the changes in ranks of integrations with respect to the different variables are justified by Fubini theorem, which is not systematically signaled in the steps of the proof. Integrating by parts the inner integral with respect to x_1 gives

$$W_F = \int\limits_{A_2\times A_3} U(a_1,x_2,x_3) F_1^{23}(a_1|X_2=x_2,X_3=x_3) dF_{23}(x_2,x_3)$$

$$- \int\limits_{A_2\times A_3} \left[\int\limits_{A_1} U_1(x_1,x_2,x_3) F_1^{23}(x_1|X_2=x_2,X_3=x_3)\, dx_1 \right] dF_{23}(x_2,x_3)$$

It is convenient to treat separately this two terms in order to present in the most economic way the proofs of propositions 5 to 7. Let call T_1 the first one and T_2 the second one. Let first evaluate T_2. We start by integrating T_2 with respect to x_3. This necessitates to separate the distributions of x_2 and x_3 by conditioning on x_2 and by using Fubini. This is done first by noticing that $dF_{23}(x_2\,,x_3) = dF_3^2(x_3\,|\,X_2=x_2)\,dF_2(x_2)$ and then by using (1) which implies that
$\int\limits_{[0,x_3]} F_1^{23}(x_1|X_2=x_2,X_3=t)\,dF_3^2(t\,|\,X_2=x_2) = F_{13}^2(x_1,x_3|X_2=x_2).$

We get

$$T_2 = -\int\limits_{A_2} \left[\int\limits_{A_1} U_1(x_1,x_2,a_3) F_{13}^2(x_1,a_3|X_2=x_2)\, dx_1 \right] dF_2(x_2) \tag{59a}$$

$$+ \int\limits_{A_2} \left[\int\limits_{A_1\times A_3} U_{13}(x_1,x_2,x_3) F_{13}^2(x_1,x_3|X_2=x_2)\, dx_1 dx_3 \right] dF_2(x_2) \tag{59b}$$

Integrating T_2 once more with respect to x_1 and denoting $H_{13}^2(x_1;x_3|X_2=x_2) := \int\limits_0^{x_1} F_{13}^2(t,x_3|X_2=x_2)dt$ gives

$$
\begin{aligned}
T_2 = & -\int\limits_{A_2} U_1(a_1, x_2, a_3) H_{13}^2(a_1; a_3 | X_2 = x_2)\, dF_2(x_2) \\
& + \int\limits_{A_2} \left[\int\limits_{A_1} U_{11}(x_1, x_2, a_3) H_{13}^2(x_1; a_3 | X_2 = x_2) dx_1 \right] dF_2(x_2) \\
& + \int\limits_{A_2} \left[\int\limits_{A_3} U_{13}(a_1, x_2, x_3) H_{13}^2(a_1; x_3 | X_2 = x_2)\, dx_3 \right] dF_2(x_2) \\
& - \int\limits_{A_2} \left[\int\limits_{A_1 \times A_3} U_{113}(x_1, x_2, x_3) H_{13}^2(x_1; x_3 | X_2 = x_2)\, dx_1 dx_3 \right] dF_2(x_2)
\end{aligned}
$$

Finally integrating T_2 with respect to x_2 and using the fact that $\int\limits_{A_2} H_{13}^2(x_1; x_3 | X_2 = x_2)\, dF_2(x_2) = H_1(x_1; x_2, x_3)$ one yields

$$T_2 = -U_1(a_1, a_2, a_3) H_1(a_1; a_2, a_3) \tag{60a}$$

$$+ \int\limits_{A_2} U_{12}(a_1, x_2, a_3) H_1(a_1; x_2, a_3) dx_2 \tag{60b}$$

$$+ \int\limits_{A_1} U_{11}(x_1, a_2, a_3) H_1(x_1; a_2, a_3) dx_1 \tag{60c}$$

$$- \int\limits_{A_1 \times A_2} U_{112}(x_1, x_2, a_3) H_1(x_1; x_2, a_3) dx_1 dx_2 \tag{60d}$$

$$+ \int\limits_{A_3} U_{13}(a_1, a_2, x_3) H_1(a_1; a_2, x_3)\, dx_3 \tag{60e}$$

$$- \int\limits_{A_2 \times A_3} U_{123}(a_1, x_2, x_3) H_1(a_1; x_2, x_3) dx_2 dx_3 \tag{60f}$$

$$- \int\limits_{A_1 \times A_3} U_{113}(x_1, a_2, x_3) H_1(x_1; a_2, x_3) dx_1 dx_3 \tag{60g}$$

$$+ \int\limits_{A_1 \times A_2 \times A_3} U_{1123}(x_1, x_2, x_3) H_1(x_1; x_2, x_3) dx_1 dx_2 dx_3 \tag{60h}$$

We now turn to get an expression for T_1. We start by integrating it by part with respect to x_2. This necessitates to separate the distributions of x_2 and x_3 by conditioning on x_3 and by using Fubini. This is done first by noticing that $dF_{23}(x_2\,, x_3) = dF_2^3(x_2 \,|\, X_3 = x_3)\, dF_3(x_3)$ and then by using (1) the fact that

$$\int\limits_0^{x_2} F_1^{23}(x_1 \,|\, X_2 = t, X_3 = x_3)\, dF_2^3(t | X_3 = x_3) = F_{12}^3(x_1, x_2 \,|\, X_3 = x_3).$$

We get

$$T_1 = \int_{A_3} U(a_1, a_2, x_3) F_{12}^3(a_1, a_2 | X_3 = x_3) dF(x_3) \tag{61}$$

$$- \int_{A_3} \int_{A_2} U_2(a_1, x_2, x_3) F_{12}^3(a_1, x_2 | X_3 = x_3)\, dx_2\, dF_3(x_3) \tag{62}$$

We now integrate (61) with respect to x_3 and (62) with respect to x_2. To be able to do it we need to know: (a) the primitive function of $F_{12}^3(a_1, a_2 | X_3 = x_3)$ with respect to x_3 and (b) the primitive function of $F_{12}^3(a_1, x_2 | X_3 = x_3)$ with respect to x_2. The first primitive is obtained as follows using (1).

$\int_{[0,x_3]} F_{12}^3(a_1, a_2 \,|\, X_3 = x_3)\; dF_3(v). = F(a_1, a_2, x_3) = F_3(x_3)$

For the second primitive. let us define

$H_{12}^3(x_{2;}\, a_1 |\, X_3 = x_3) := \int_0^{x_2} F_{12}^3(a_1,\, t \,|\, X_3 = x_3)\, dt$

Therefore,

$$T_1 = U(a_1, a_2, a_3) F(a_1, a_2, a_3) \tag{63}$$

$$- \int_{A_3} U_3(a_1, a_2, x_3)\, F_3(x_3)\, dx_3 \tag{64}$$

$$- \int_{A_3} U_2(a_1, a_2, x_3)\, H_{12}^3(a_2\,; a_1 |\, X_3 = x_3)\, dF_3(x_3) \tag{65}$$

$$+ \int_{A_3} \int_{A_2} U_{22}(a_1, x_2, x_3)\, H_{12}^3(x_2\,; a_1 |\, X_3 = x_3)\, dx_2\, dF_3(x_3) \tag{66}$$

Finally we integrate the three last terms of the RHS of the above expression with respect to x_3. This implies to define

$\int_0^{x_3} H_{12}^3(x_2\,; a_1 |\, X_3 = x_3)\, dF_3(x_3) = \int_0^{x_3} \int_0^{x_2} F_{12}^3(a_1, t \,|\, X_3 = x_3)\, dt dF_3(x_3) =$

$\int_0^{x_2} F(a_1, t, x_3\,)\, dt := H_2(x_2; a_1, x_3)$

We finally obtain

$$T_1 = U(a_1,a_2,a_3)F(a_1,a_2,a_3) \tag{67a}$$

$$-U_3(a_1,a_2,a_3)\,H_3(a_3) + \int\limits_{A_3} U_{33}(a_1,a_2,x_3)\,H_3(x_3)\,dx_3 \tag{67b}$$

$$-\int\limits_{A_3} U_2(a_1,a_2,x_3)H_2(a_2) \;+ \int\limits_{A_3} U_{23}(a_1,a_2,x_3)\,H_2(a_2;a_1,x_3)\,dx_3 \tag{67c}$$

$$+\int\limits_{A_2} U_{22}(a_1,x_2,a_3)\,H_2(x_2)\,dx_2 \tag{67d}$$

$$-\int\limits_{A_3}\int\limits_{A_2} U_{223}(a_1,x_2,x_3)H_2(x_2;a_1,x_3)\,dx_2\,dx_3 \tag{67e}$$

Therefore the expression for the welfare associated to F is given by

$$W_F = U(a_1,a_2,a_3)F(a_1,a_2,a_3) \tag{68a}$$

$$-U_3(a_1,a_2,a_3)\,H_3(a_3) + \int\limits_{A_3} U_{33}(a_1,a_2,x_3)\,H_3(x_3)\,dx_3 \tag{68b}$$

$$-\int\limits_{A_3} U_2(a_1,a_2,x_3)H_2(a_2) \;+ \int\limits_{A_3} U_{23}(a_1,a_2,x_3)\,H_2(a_2;a_1,x_3)\,dx_3 \tag{68c}$$

$$+\int\limits_{A_2} U_{22}(a_1,x_2,a_3)\,H_2(x_2)\,dx_2 \tag{68d}$$

$$-\int\limits_{A_3}\int\limits_{A_2} U_{223}(a_1,x_2,x_3)H_2(x_2;a_1,x_3)\,dx_2\,dx_3 \tag{68e}$$

$$-U_1(a_1,a_2,a_3)H_1(a_1;a_2,a_3) + \int\limits_{A_3} U_{13}(a_1,a_2,x_3)H_1(a_1;a_2,x_3)dx_3 \tag{68f}$$

$$+\int\limits_{A_2} U_{12}(a_1,x_2,a_3)H_1(a_1;x_2,a_3)dx_2 - \int\limits_{A_2}\int\limits_{A_3} U_{123}(a_1,x_2,x_3)H_1(a_1;x_2,x_3)dx_2dx_3 \tag{68g}$$

$$+\int\limits_{A_1} U_{11}(x_1,a_2,a_3)H_1(x_1;a_2,a_3)\,dx_1 - \int\limits_{A_3}\int\limits_{A_1} U_{113}(x_1,a_2,x_3)H_1(x_1;a_2,x_3)dx_1dx_3 \tag{68h}$$

$$-\int\limits_{A_2}\int\limits_{A_1} U_{112}(x_1,x_2,a_3)H_1(x_1;x_2,a_3)dx_1\,dx_2 \tag{68i}$$

$$+\int\limits_{A_3}\int\limits_{A_2}\int\limits_{A_1} U_{1123}(x_1,x_2,x_3)H_1(x_1;x_2,x_3)dx_1dx_2dx_3 \tag{68j}$$

Taking into account the assumptions, we obtain for the change in welfare

$$\Delta W_U = -U_3(a_1,a_2,a_3)\,\Delta H_3(a_3) + \int\limits_{A_3} U_{33}(a_1,a_2,x_3)\,\Delta H_3(x_3)\,dx_3 \tag{69a}$$

$$-\int\limits_{A_3} U_2(a_1,a_2,x_3)\Delta H_2(a_2) \ + \int\limits_{A_3} U_{23}(a_1,a_2,x_3)\,\Delta H_2(a_2;a_1,x_3)\,dx_3 \tag{69b}$$

$$+\int\limits_{A_2} U_{22}(a_1,x_2,a_3)\,\Delta H_2(x_2)\,dx_2 \tag{69c}$$

$$-\int\limits_{A_3}\int\limits_{A_2} U_{223}(a_1,x_2,x_3)\Delta H_2(x_2;a_1,x_3)\,dx_2\,dx_3 \tag{69d}$$

$$-U_1(a_1,a_2,a_3)\Delta H_1(a_1) + \int\limits_{A_3} U_{13}(a_1,a_2,x_3)\Delta H_1(a_1;a_2,x_3)dx_3 \tag{69e}$$

$$+\int\limits_{A_2} U_{12}(a_1,x_2,a_3)\Delta H_1(a_1;x_2,a_3)dx_2 - \int\limits_{A_2}\int\limits_{A_3} U_{123}(a_1,x_{2,}x_3)\Delta H_1(a_1;x_2,x_3)dx_2dx_3 \tag{69f}$$

$$+\int\limits_{A_1} U_{11}(x_1,a_2,a_3)\Delta H_1(x_1)\,dx_1 - \int\limits_{A_3}\int\limits_{A_1} U_{113}(x_1,a_2,x_3)\Delta H_1(x_1;a_2,x_3)dx_1dx_3 \tag{69g}$$

$$-\int\limits_{A_2}\int\limits_{A_1} U_{112}(x_1,x_2,a_3)\Delta H_1(x_1;x_2,a_3)dx_1\,dx_2 \tag{69h}$$

The conclusion follows.

Necessity.

Let us suppose that condition B_5 is not fullfilled. Then there exists $x_3^* \in X_2$ such that $\Delta H_3(x_3^*) > 0$.

Let us define the utility function V_{B_5} as follows.

$$\forall\ x_1 \in X_1, \forall\ x_2 \in X_2, , V_{B_5}(x_1,x_2,x_3) = x_3 - x_3^* \ \forall\ x_3 \le x_3^* \text{ and } V_{B_5}(x_1,x_2) = 0,\ \forall\ x_3 > x_3^*.$$

It is readily shown that $\triangle W_{V_{B_5}} = -\Delta H_3(x_3^*)$. Clearly $V_{B_{B_5}}$ does not belong to $\mathcal{U}^5$, but it is possible to find an approximation of $V_{B_{B_5}}$ which belongs to $\mathcal{U}^5$.

Let us supposed that condition C_5 is not fullfilled. Then it must be true that there exists $(x_2^*, x_3^*\ ) \in X_2$ xX_3 such that $\Delta H_2(x_2^*;a_{_1},x_3^*) > 0$.Let us define the utility function V_{D_5} such that

$$\forall\ x_1 \le a_1,\ \forall\ x_2 \le x_2^*\ ,\ \forall x_3 \le x_3^*, \qquad V_{D_5}(x_1,x_{2,}x_3) = x_2 - x_2^*, \tag{70}$$

$$\text{otherwise} \qquad V_{D_5}(x_1,x_{2,}x_3) = 0, \tag{71}$$

It is readily shown that $\triangle W_{V_{D_5}} = -\Delta H_2(x_2^*;a_{_1},x_3^*)$. Clearly V_{D_5} does not belong to $\mathcal{U}^5$ but it is possible to find an approximation of V_{D_5} which belongs to $\mathcal{U}^5$.

Let us supposed that condition D_5 is not fullfilled. Then there exists $(x_1^*, x_2^*) \in X_1$ x X_2 such that $\Delta H_1(x_1^*;x_2^*,a_3) > 0$. Let us define the utility function V_{D_5} such that

$$\forall\ x_1 \le x_1^*,\ \forall\ x_2 \le x_2^*, \text{and for } x_3 \le a_3 \qquad V_{D_5}(x_1,x_{2,}x_3) = x_1 - x_1^*, \tag{72}$$

$$\text{otherwise} \qquad V_{D_5}(x_1,x_{2,}x_3) = 0, \tag{73}$$

It is readily shown that $\triangle W_{V_{D_5}} = -\Delta H_1(x_1^*; x_2^*, a_3)$. Clearly V_{D_5} does not belong to $\mathcal{U}^5$ but it is possible to find an approximation of V_{D_5} which belongs to $\mathcal{U}^5$.

Let us supposed that condition E_5 is not fulfilled. Then it must be true that there exists $(x_1^*, x_3^*) \in X_1$ xX_3 such that $\Delta H_1(x_1; a_2, x_3) > 0$.Let us define the utility function V_{E_5} such that

$$\forall\ x_1 \leq x_1^*, \forall\ x_2 \leq a_2\ , \forall x_3 \leq x_3^*, \qquad V_{E_5}(x_1, x_{2,} x_3) = x_1 - x_1^*, \tag{74}$$

$$\text{otherwise} \qquad V_{E_5}(x_1, x_{2,} x_3) = 0, \tag{75}$$

It is readily shown that $\triangle W_{V_{E_5}} = -\Delta H_1(x_1; a_2, x_3)$. Clearly V_{E_5} does not belong to $\mathcal{U}^5$ but it is possible to find an approximation of V_{E_5} which belongs to $\mathcal{U}^5$. The proof for the necessity of F_5 is similar.

Proof of Proposition 3.3:

Sufficiency. The expression for T_2 in equation (59) taking into account that $U_{13} = 0$ reduces to :

$$T_2 = -\int_{A_2} \left[\int_{A_1} U_1(x_1, x_2, a_3) F_{13}^2(x_1, a_3 | X_2 = x_2)\, dx_1 \right] dF_2(x_2)$$

Performing the same integrations than in Proposition 5's proof we get

$$\begin{aligned} T_2 &= -U_1(a_1, a_2, a_3) H_1(a_1; a_2, a_3) \\ &+ \int_{A_2} U_{12}(a_1, x_2, a_3) H_1(a_1; x_2, a_3) dx_2 \\ &+ \int_{A_1} U_{11}(x_1, a_2, a_3) H_1(x_1; a_2, a_3) dx_1 \\ &- \int_{A_1 \times A_2} U_{112}(x_1, x_2, a_3) H_1(x_1; x_2, a_3) dx_1 dx_2 \end{aligned}$$

The expression for T_1 in equation (67) remains valid. Therefore the expression for the change in welfare becomes

$$\Delta W_U = -U_3(a_1, a_2, a_3)\,\Delta H_3(a_3) + \int_{A_3} U_{33}(a_1, a_2, x_3)\,\Delta H_3(x_3)\,dx_3 \tag{76a}$$

$$-\int_{A_3} U_2(a_1, a_2, x_3)\Delta H_2(a_2) \; + \int_{A_3} U_{23}(a_1, a_2, x_3)\,\Delta H_2(a_2; a_1, x_3)\,dx_3 \tag{76b}$$

$$+\int_{A_2} U_{22}(a_1, x_2, a_3)\,\Delta H_2(x_2)\,dx_2 \tag{76c}$$

$$-\int_{A_3}\int_{A_2} U_{223}(a_1, x_2, x_3)\Delta H_2(x_2; a_1, x_3)\,dx_2\,dx_3 \tag{76d}$$

$$-U_1(a_1, a_2, a_3)\Delta H_1(a_1) \tag{76e}$$

$$+\int_{A_2} U_{12}(a_1, x_2, a_3)\Delta H_1(a_1; x_2, a_3)dx_2 \tag{76f}$$

$$+\int_{A_1} U_{11}(x_1, a_2, a_3)\Delta H_1(x_1)\,dx_1 \tag{76g}$$

$$-\int_{A_2}\int_{A_1} U_{112}(x_1, x_2, a_3)\Delta H_1(x_1; x_2, a_3)dx_1\,dx_2 \tag{76h}$$

Necessity. See proof of proposition 3.1.

Proof of Proposition 3.5:

Sufficiency. The expression for T_2 obtained in Proposition 5's proof remains valid. To compute T_1 we change the integration path. We start by integrating it by part with respect to x_3. This necessitates to separate the distributions of x_3 and x_2 by conditioning on x_2 and by using Fubini.

We get

$$T_1 = \int_{A_2} U(a_1, x_2, a_3)F_{13}^2(a_1, a_3|X_2 = x_2)\,dF_2(x_2) \tag{77}$$

$$-\int_{A_2}\left[\int_{A_3} U_3(a_1, x_2, x_3)F_{13}^2(a_1, x_3|X_2 = x_2)\,dx_3\right] dF_2(x_2) \tag{78}$$

We now integrate the RHS of the above expression with respect to x_2. We obtain

$$
\begin{aligned}
T_1 &= U(a_1,a_2,a_3)F(a_1,a_2,a_3) \\
&- \int_{A_2} U_2(a_1,x_2,a_3)F(a_1,x_2,a_3)\,dx_2 \\
&- \int_{A_3} U_3(a_1,a_2,x_3)F(a_1,a_2,x_3)dx_3 \\
&+ \int_{A_2\times A_3} U_{32}(a_1,x_2,x_3)F(a_1,x_2,x_3)\,dx_2dx_3
\end{aligned}
$$

Since $U_{32}=0$, the last term vanishes. Finally we integrate the second term of the above expression with respect to x_2 and the third term with respect to x_3. One yields

$$
\begin{aligned}
T_1 &= U(a_1,a_2,a_3)F(a_1,a_2,a_3) \\
&-U_2(a_1,a_2,a_3)H_2(a_2;a_1,a_3) \\
&+ \int_{A_2} U_{22}(a_1,x_2,a_3)H_2(x_2;a_1,a_3)\,dx_2 \\
&-U_3(a_1,a_2,a_3)H_3(a_3;a_1,a_2) \\
&+ \int_{A_3} U_{33}(a_1,a_2,x_3)H_3(x_3;a_1,a_2)dx_3
\end{aligned}
$$

Therefore the expression for the change in welfare becomes

$$
\begin{aligned}
\Delta W_U &= -U_2(a_1,a_2,a_3)\Delta H_2(a_2) \\
&+ \int_{A_2} U_{22}(a_1,x_2,a_3)\Delta H_2(x_2)\,dx_2 \\
&-U_3(a_1,a_2,a_3)\Delta H_3(a_3) \\
&+ \int_{A_3} U_{33}(a_1,a_2,x_3)\Delta H_3(x_3)dx_3 \\
&-U_1(a_1,a_2,a_3)\Delta H_1(a_1) + \int_{A_3} U_{13}(a_1,a_2,x_3)\Delta H_1(a_1;a_2,x_3)dx_3 \\
&+ \int_{A_2} U_{12}(a_1,x_2,a_3)\Delta H_1(a_1;x_2,a_3)dx_2 - \int_{A_2}\int_{A_3} U_{123}(a_1,x_2,x_3)\Delta H_1(a_1;x_2,x_3)dx_2dx_3 \\
&+ \int_{A_1} U_{11}(x_1,a_2,a_3)\Delta H_1(x_1)\,dx_1 - \int_{A_3}\int_{A_1} U_{113}(x_1,a_2,x_3)\Delta H_1(x_1;a_2,x_3)dx_1dx_3 \\
&- \int_{A_2}\int_{A_1} U_{112}(x_1,x_2,a_3)\Delta H_1(x_1;x_2,a_3)dx_1\,dx_2 \\
&+ \int_{A_3}\int_{A_2}\int_{A_1} U_{1123}(x_1,x_2,x_3)\Delta H_1(x_1;x_2,x_3)dx_1dx_2dx_3
\end{aligned}
$$

Since the utility function has the form (21), $U_{123} = 0$, and $U_{1123} = 0$. The conclusion follows.

Necessity. See proof of proposition 3.1.